Dynamic Budget Throttling in Repeated Second-Price Auctions

Zhaohua Chen^{*†}

Chuyue Tang[§]

Chang Wang^{*‡} Zheng Cai[¶]

^{*‡} Qian Wang^{*†} Yukun Ren[¶] Zhuming Shi[‡] Xiaotie Deng[∥]

Yuqi Pan[‡]

Zhihua Zhu ¶

July 18, 2022

Abstract

Throttling is one of the most popular budget control methods in today's online advertising markets. When a budget-constrained advertiser employs throttling, she can choose whether or not to participate in an auction after the advertising platform recommends a bid. This paper focuses on the dynamic budget throttling process in repeated second-price auctions from a theoretical view. An essential feature of the underlying problem is that the advertiser does not know the distribution of the highest competing bid upon entering the market. To model the difficulty of eliminating such uncertainty, we consider two different information structures. The advertiser could obtain the highest competing bid in each round with full-information feedback. Meanwhile, with partial information feedback, the advertiser could only have access to the highest competing bid in the auctions she participates in. We propose the OGD-CB algorithm, which involves simultaneous distribution learning and revenue optimization. In both settings, we demonstrate that this algorithm guarantees an $O(\sqrt{T \log T})$ regret with probability 1 - O(1/T) relative to the fluid adaptive throttling benchmark. By proving a lower bound of $\Omega(\sqrt{T})$ on the minimal regret for even the hindsight optimum, we establish the near optimality of our algorithm. Finally, we compare the fluid optimum of throttling to that of pacing, another widely adopted budget control method. The numerical relationship of these benchmarks sheds new light on the understanding of different online algorithms for revenue maximization under budget constraints.

^{*}These authors contributed equally to this work.

[†]CFCS, School of Computer Science, Peking University. Email: {chenzhaohua, charlie}@pku.edu.cn

[‡]School of Electronics Engineering and Computer Science, Peking University. Email: {wchang, pyq0419, shizhuming}@pku.edu.cn

[§]Yuanpei College, Peking University. Email: 2000017814@stu.pku.edu.cn

[¶]Tencent Technology (Shenzhen) Co., Ltd. Email: {zhengcai, rickyren, zhihuazhu}@tencent.com

^ICFCS, School of Computer Science, Peking University and CMAR, Institute for Artificial Intelligence, Peking University. Email: xiaotie@pku.edu.cn

1 Introduction

The online advertising market has been enjoying prominent growth in recent years. The booming of new social media forms, e.g., short video platforms, also accelerates the enlargement of such a giant market. When a user submits an ad query to the market, an auction is held among all advertisers who are interested in the query, and the winner gets the chance to display her ad. In practice, due to the enormous market volume, it is common for an advertiser to set a budget to control the expenditure across a specific time range. Correspondingly, advertising platforms offer advertisers multiple kinds of budget control methods.

This paper studies one of these methods, called *throttling* (a.k.a. probabilistic pacing), which is widely adopted by major advertising platforms including Facebook [29], Google [39], LinkedIn [4] and Yahoo! [51]. Under throttling, an advertiser's accumulated payment is controlled by being excluded from a fraction of auctions throughout the total period (e.g., a day, a week, or a month). Compared to other budget management strategies [15], e.g., (multiplicative) pacing (a.k.a. bid-shading), an essential feature of throttling is that an advertiser's bid is not modified in any auction instance. In real-life, such an option becomes a prevalent choice, particularly for those small advertisers, e.g., low-cost app producers. A primary reason is that these advertisers, with an incomplete enterprise structure, may not be able to collect enough information on the whole market to set adaptive bids for heterogeneous queries. On the other hand, advertising platforms could acquire more knowledge on the value of an ad display with more historical data on the market. As a result, many advertisers would leave the rights to the platform to set the bid in each auction, and these advertisers would only control whether to participate in an auction given the bid. Such a phenomenon highlights the importance of throttling as a popular budget control method.

While previous works mainly focused on the throttling strategy from the practical side [4,39,51] or in an equilibrium view [24], this paper aims to find a dynamic throttling strategy for an advertiser in repeated second-price auctions with *theoretically* guaranteed performance. Technically, this problem belongs to the class of contextual bandits problems with knapsack constraints (CBwK) [7, 11]. In particular, two problem families with fruitful applications lie in this range, namely dynamic resource-constrained reward collection (DRC²) problem [12] and online resource allocation problem [17]. (See a further review on related literature in Section 1.2.) Nevertheless, our problem critically and inevitably differs from the CBwK prototype and the two concrete meta-problems. We summarize the differences in Table 1.

More concretely, with almost tight efficiency in the solutions, our problem generalizes the above three problems in three aspects: (a) The size of the context set in our problem could be arbitrary from finite to continuum; (b) The reward and the cost are *stochastic* to the agent (the advertiser) even after she knows the context (her bid) and makes the decision either to enter or not to; and (c) the distributions of the reward and the cost are *unknown* to the agent given the context and the agent's decision. For (a), we mention that other works that require a finite context space could fit into the general case by conducting a discretization on the context set. However, such a procedure would inescapably increase the regret of the algorithms asymptotically. Meanwhile, (b) and (c) imply that our problem requires a procedure that can unify learning the unknown distributions and optimizing the cumulative rewards. We should emphasize that these two points are common sense in the context of ad auction. In each second-price auction instance, an advertiser's reward and cost are determined collaboratively by her bid and the highest competing bid. Since there are a large number of advertisers in the market simultaneously, it is reasonable to take the mean-field approximation [13] and assume that the latter is drawn i.i.d. from some distribution. Nevertheless. such a distribution is intricately correlated with multiple factors, including the market size, the preference of competing advertisers, and their active time in the market. Consequently, it is almost

		Given Context & Action:		
Contextual Bandits Problems with Knapsacks	Size of Context Set	Reward & Cost Stochastic/Fixed?	Distributions of Reward & Cost Known?	
Prototype [7,11]	Finite	Stochastic	Unknown	
Dynamic Resource Constrained Reward Collection [12]	Finite	Stochastic	Known	
Online Resource Allocation [17]	Arbitrary	Fixed	/	
Dynamic Throttling in Repeated SPA	Arbitrary	Stochastic	Unknown	

Table 1: Difference in the setting of the problem considered in this work and other widely-considered contextual bandit problems with knapsacks.

impossible for a single advertiser to know this distribution *ex-ante*. Regarding the intrinsic features of our problem, our result could provide a new heuristic for simultaneous learning and optimization to the abundant correlated literature.

1.1 Our Contributions and Techniques

To settle the process of learning the distribution of the highest competing bid, we consider two information structures in order, with the increasing difficulty of getting a sample. With full information feedback, the advertiser could obtain the highest competing bid in each round. However, such an assumption could be too strong for practice. Therefore, we further consider the partial information feedback structure, in which the advertiser can acquire the highest competing bid only when she participates in an auction. We mention that the latter feedback model is realizable in the industry if the platform announces the highest bid to all participants after an auction. To measure the performance of an online throttling strategy, we compare the reward brought by a strategy with the fluid adaptive throttling optimum (1). Such an optimum captures the fluid reward of the optimal adaptive throttling strategy when the budget constraint is met in expectation. We mention that computing such a benchmark requires knowledge of the distributions of the advertiser's bid and the highest competing bid, and is impossible for the advertiser *a priori*.

The OGD-CB algorithm and analysis. In this paper, we propose the OGD-CB algorithm, which guarantees an $O(\sqrt{T \log T})$ regret with probability 1 - O(1/T) with either full/partial information feedback (Section 3). In each round, the algorithm is composed of three parts: (a) update the estimates on the target distribution according to previous samples; (b) decide the action according to the estimates and the dual variable; (c) execute an online gradient descent (OGD) procedure on the dual variable. We now explain how the algorithm works in intuition.

Generally, the optimal fluid adaptive throttling strategy follows a threshold structure captured by the optimal dual solution. Nevertheless, such a solution cannot be computed directly by the advertiser in advance without knowledge of the competing distribution. We solve this issue by introducing the dual variable, which aims to approach the optimal dual parameter via an OGD procedure. Correspondingly, the decision in each round is made using the approximated dual variable. With such a heuristic, we can further prove that the whole procedure does not end too early. Intuitively, when the cost is low (or high, respectively) in a round, the dual variable would correspondingly decrease (increase), rendering a higher (lower) probability for the advertiser to participate in the upcoming rounds. Such an adjustment ensures that the total budget is used up smoothly around the target expenditure rate. We should mention that a similar methodology has already been adopted in previous works, e.g., [17].

A critical subject in our problem is estimating each round's expected reward and cost after seeing the bid. However, the advertiser does not know the distribution of the highest competing bid, so she has to learn an empirical distribution via previous samples. Meanwhile, such a learning process should not interfere with the optimizing procedure. For this part, we adopt the Dvoretzky–Kiefer–Wolfowitz (DKW) inequality [45] and form a confidence bound (CB) on the estimation results. With full information feedback, the number of samples the advertiser sees increases each round. As a result, the sum of the lengths of confidence intervals across all rounds can be bounded within $O(\sqrt{T \log T})$. Besides, with partial information feedback, it is important that the advertiser participates with a high frequency. Or else, the learning process will not converge quickly, and the algorithm's performance will have no guarantee. Via a novel argument on the dynamic change of the dual variable, we can show that our OGD-CB algorithm guarantees a *constant* entering frequency without any adjustment. Hence, the $O(\sqrt{T \log T})$ bound still holds for our method even with only partial information feedback.

Our algorithm has two main advantages. First, our algorithm is computationally friendly. In each round, the update process of the dual variable only takes a constant time. Second, compared with other algorithms, our solution does not rely on any particular properties of the distribution of the highest competing bid. (See Section 1.2 for a review in literature.) Specifically, our solution does not require the (interim) reward/cost function to be linear, convex/concave, or even continuous. In particular, our solution even works for discrete distributions.

Lower bound on the regret of online throttling. We continue to explore the lower bound of the regret of any online throttling strategy compared with the fluid adaptive throttling optimum (Section 4). Our result shows that an $\Omega(\sqrt{T})$ regret would be unavoidable for online throttling algorithms under certain problem instances. This result is an extension of the results in [9,21,49]. Consequently, we argue that our OGD-CB algorithm, which guarantees an $O(\sqrt{T \log T})$ regret with high probability, reaches near optimality.

Comparison between pacing and throttling. We also compare the throttling strategy with another widely adopted budget control strategy, known as pacing (Section 5). Notably, we focus on the fluid optima of these two strategies. Our results show that these two optima could generally have arbitrary numerical order. Nevertheless, when the optimal fluid pacing strategy exactly depletes the budget, the corresponding reward is no less than the fluid adaptive throttling benchmark. In Appendix A, we conduct a further systematic comparison of these two optima with the *hindsight optimum*, which is also usually applied as the benchmark for online pacing algorithms [14, 22]. We demonstrate that the numerical relationship of these benchmarks is volatile to the problem instance (Lemma A.6, Lemma A.7). Our results could provide further insights into comparing different online throttling and pacing algorithms.

1.2 Additional Related Works

In this part, we will first review two popular budget management strategies in repeated auctions: throttling and pacing. Further, we will establish our problem's relationship with works closely connected in technique.

Previous works on dynamic throttling mainly stand on the experimental side. Among them, [39] considers the concept of fair allocation in generalized second-price (GSP) auctions, in which they define an optimal throttling algorithm for different objectives. [4] also focuses on GSP auctions from a buyer's side and implements their algorithm in LinkedIn's ad serving system. [51] evaluates a practical online throttling algorithm on the demand-side platform. [23] considers the regret-free allocation for buyers' ROI, and shows that such a heuristic outperforms the random throttling strategy for buyers. On the other hand, [24] focuses on the market equilibrium computationally when all buyers simultaneously follow a random throttling strategy. Compared to our work, these works do not engage in a theoretical view of the dynamics of a throttling strategy. Some other works consider a similar problem for Internet keyword search, known as the AdWords problem [46, 47] in the framework of online matching. As we will come to in the following paragraphs, this problem is different from our problem in multiple ways.

Pacing is also a well-studied budget control strategy, in which an advertiser shades her value by a constant factor on her bid. Existing works study such strategy from either a dynamic view [14,19,22,32,35,43] or an equilibrium view [13,26,27]. Within these works, [14] is highly correlated with our solution, which also takes a sight from the dual space. Nevertheless, the analysis of their algorithm relies on the continuity of the distribution function, which is not necessary for our algorithm. Some papers compare different budget control methods and explore their relationships on the standpoint of equilibrium [15,16,25]. In particular, [15] shows that in the symmetric system equilibrium, throttling brings a higher profit for the seller than pacing under certain assumptions. Some of our results extend such a comparison between throttling and pacing on a buyer's side.

On the technical side, our problem belongs to the contextual bandits problems with knapsacks (CBwK) [6,7,11,50]. However, the specific problems considered in those works are different from our problem in multiple aspects. Particularly, the regret bound of the algorithms for the prototype problem requires a finite context space [7, 11]. [6] assumes that the expectations of reward and cost are linearly correlated with the context. Further, [50] supposes that the cost of an action is fixed given the context. In contrast, our problem allows an infinite context space, and the distributions of reward and cost could be arbitrary. We also compare our problem with another two sets of problems in CBwK. The first family is dynamic resource-constrained reward collection (DRC²) problems, which include topics like bidding in repeated auctions [1, 13, 31], dynamic knapsack [10, 41, 42], multi-secretary [9, 18, 20], network revenue management [21, 33, 37], online matching [30, 40, 49], and order fulfillment [2, 3, 38], etc. Readers can find a comprehensive survey on these problems in [12]. This family mainly differs from our problem in that in these problems, the distributions of reward and cost are known given the context and action. However, in our problem, such distributions are unknown to the advertiser in advance. Meanwhile, these problems suppose that the context set is finite. The second family that is close to our problem is online allocation problems [5, 8, 17, 28, 34, 44, 48]. Nonetheless, this problem family explicitly supposes fixed reward and cost given a context and action pair, which is not the case for our problem.

Organization. The rest of this work is organized as follows. We introduce the problem model in Section 2. In section 3, we introduce the OGD-CB algorithm and give a regret bound of the algorithm under both full and partial information feedback settings. We present the lower bound of the problem in Section 4. Further, we compare fluid throttling and pacing in Section 5. We conclude this work in Section 6.

2 Model

Basic Settings. In this work, we consider the repeated second-price auction market, in which an advertiser with a budget constraint competes with other advertisers. To match the more prevalent terminology in literature, we call the "advertiser" the "buyer" in the remaining parts. Assume there are T rounds of auctions in total, and in each round $t \in [T] := \{1, 2, \dots, T\}$, an item is to be sold to a buyer via a second-price auction. Here, we suppose that $T \gg 1$.

This work stands from the viewpoint of a fixed buyer. In each round $t \in [T]$, she obtains a personal value on the item, which we denote by v_t . In real-life, v_t captures the recommended bid from the seller (an advertising platform). We suppose that v_t is drawn independently from a distribution F with support [0, 1]. We use v to represent the vector of buyer's value across all Trounds, i.e., $v := (v_t)_{t \in [T]}$. On the other hand, we suppose that in round t, the highest competing bid for the buyer is p_t , which is independently sampled from a distribution G with support [0, 1]. Such an assumption comes from the mean-field approximation [13] and is commonly adopted in the literature. Similarly, we let $p := (p_t)_{t \in [T]}$. In this work, we suppose that both F and G are unknown to the buyer.

We assume the buyer has a total budget of B across all T rounds, with the maximum average expenditure being $\rho := B/T$ per round. In this paper, we suppose that ρ is a constant. This assumption comes from the practice in which the buyer is always asked to set a budget for a fixed period, with relatively fixed rounds of auctions. To ensure that her total payment is within the budget, in each round t, the buyer makes a decision $x_t \in \{0, 1\}$ after seeing the value v_t . Here, the binary choice of a buyer reflects the nature of throttling, with $x_t = 1$ representing entering the auction, and $x_t = 0$ standing for saving the budget and not entering the auction. After the decision, the buyer gets a revenue of x_tr_t and a cost of x_tc_t in this round, in which r_t and c_t are defined respectively as:

$$r_t = (v_t - p_t)^+, \quad c_t = p_t \mathbf{1}[v_t \ge p_t].$$

Here, $(v_t - p_t)^+$ stands for the positive part of $(v_t - p_t)$ in the formula of r_t . The buyer gets positive revenue and cost in round t only when she chooses to "enter" (i.e., $x_t = 1$) and wins (i.e., $v_t \ge p_t$), in which case her cost is p_t and her revenue is $v_t - p_t$ for a second-price auction. Again, we mention here that in the literature of throttling, it is supposed that the buyer bids truthfully (follow the seller's recommendation) as long as she enters an auction.

Information Structure. Practically, the condition under which the buyer can observe p_t of round t is crucial to her strategy. Intuitively, the easier the buyer can view p_t , the more likely the buyer can obtain a throttling algorithm with a better revenue guarantee. Aiming at this point, we consider two different settings on the information structure in this work:

- 1. [Full information feedback.] The buyer sees p_t at the end of any round t.
- 2. [Partial information feedback.] The buyer sees p_t at the end of round t only when her decision is $x_t = 1$ in this round, i.e., chooses to participate in the auction.

Compared to the full information feedback model, it is certain that the partial information feedback model is a more difficult one to deal with, since the buyer accesses less information.

We now define the history the buyer can see at the start of each round in these two models. In the full information feedback model, we use $\mathcal{H}_t^{\mathrm{F}}$ to denote the view of the buyer at the start of round t:

$$\mathcal{H}_t^{\mathrm{F}} := (v_\tau, x_\tau, p_\tau)_{1 \le \tau < t},$$

since in this setting, at the end of each round τ , p_{τ} is always accessible to the buyer, and r_{τ} and c_{τ} can be deduced from v_{τ} , x_{τ} and p_{τ} . Meanwhile, for the partial information feedback model, since p_{τ} is revealed to the buyer if and only if $x_{\tau} = 1$, we define $\mathcal{H}_t^{\mathrm{P}}$ correspondingly as:

$$\mathcal{H}_t^{\mathbf{P}} := (v_\tau, x_\tau, x_\tau p_\tau)_{1 \le \tau < t}.$$

Notice that p_t cannot be reversely computed from x_t and $x_t p_t$ when $x_t = 0$. Similarly, r_{τ} and c_{τ} can be derived with v_{τ} , x_{τ} and $x_{\tau} p_{\tau}$ with partial information feedback.

The throttling strategy. With the above notations, we now formally define the throttling strategy of the buyer. We generally use \mathcal{H}_t to denote the history that the buyer can access at the start of round t, which is $\mathcal{H}_t^{\mathrm{F}}$ in the full information feedback model and $\mathcal{H}_t^{\mathrm{P}}$ in the partial information feedback model. Further, we use $\widetilde{\mathcal{H}}_t$ to denote the buyer's view when she sees her value at round t and is ready to make the decision. Specifically,

$$\mathcal{H}_t := (\mathcal{H}_t, v_t, \gamma_t)$$
 .

Here, $\gamma_t \in \Gamma$ is drawn from a probability space to depict the possible randomness that is applied to compute x_t . We further define $\gamma = (\gamma_t)_{t \in [T]}$. We use $\beta_t : [0, 1]^{3t+1} \times \Gamma \to \{0, 1\}$ to represent the buyer's throttling strategy in round t, for which

$$x_t = \beta_t(\mathcal{H}_t).$$

As a result, $\beta := (\beta_t)_{t \in [T]}$ includes the buyer's specific strategy across all T rounds and denotes her overall throttling strategy. Given the inputs $\boldsymbol{v}, \boldsymbol{p}$ and randomness $\boldsymbol{\gamma}$, we denote by $T_0^{\beta}(\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\gamma}) \leq T$ the ending time of strategy β , that is, the last round with $x_t = 1$. For succinctness, we abbreviate this term by T_0 when there is no ambiguity. Therefore, the total revenue of β with inputs \boldsymbol{v} and \boldsymbol{p} is

$$U^{eta}(oldsymbol{v},oldsymbol{p},oldsymbol{\gamma}) := \sum_{t=1}^{T_0} x_t r_t.$$

Benchmark and Regret. In this work, we consider the fluid adaptive throttling benchmark, which is the optimal expected revenue of any random strategy given the value without exceeding the budget in expectation. Specifically,

$$OPT := T \cdot \max_{\pi: [0,1] \to [0,1]} \mathbb{E}_{v \sim F, p \sim G} \left[\pi(v) \cdot (v-p)^+ \right], \quad \text{s.t.} \quad \mathbb{E}_{v \sim F, p \sim G} \left[\pi(v) \cdot p\mathbf{1}[v \ge p] \right] \le \rho.$$
(1)

We have three remarks on the benchmark we defined above: (a) The benchmark sets a different throttling parameter for each possible value of the buyer, which captures the buyer's sight, as she can only see the item's value before making a decision. (b) Computing the benchmark requires direct knowledge of the distributions F and G, which is not the case in reality. Therefore, it is likely that this benchmark is an upper bound on the expected revenue of any online throttling strategy. In fact, we show in Section 4 that there is some instance under which even the best offline/hindsight throttling strategy has an $O(\sqrt{T})$ gap to the benchmark. (c) To further demonstrate its reasonability, we will compare the benchmark with another fixed throttling benchmark in Appendix A.

Consequently, we can define the regret of strategy β facing OPT given $\boldsymbol{v}, \boldsymbol{p}$ and randomness $\boldsymbol{\gamma}$, that is,

$$Reg^{\beta}(\boldsymbol{v},\boldsymbol{p},\boldsymbol{\gamma}) := OPT - U^{\beta}(\boldsymbol{v},\boldsymbol{p},\boldsymbol{\gamma}).$$

Clearly, our goal is to design a strategy β such that $Reg^{\beta}(\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\gamma})$ is small with high probability on samples $(\boldsymbol{v}, \boldsymbol{p})$ and randomness $\boldsymbol{\gamma}$. Nevertheless, all the strategies we consider in this work does not involve any kind of randomness besides the input. As a result, we ignore the term $\boldsymbol{\gamma}$ in all formulas.

3 The OGD-CB Algorithm

In this section, we introduce an online throttling strategy known as OGD-CB. We obtain an upper bound of $O(\sqrt{T \log T})$ on the regret of the OGD-CB algorithm with either full information or partial information feedback. All omitted proofs in this section can be found in Appendix B.

3.1 The Algorithm

Our algorithm is depicted in Algorithm 1. The algorithm first conducts an one-round exploration to make an appropriate initialization (Line 3–Line 6). After observing the value in each of the following rounds, the algorithm first gives estimates of the expected revenue and cost based on the observed history prices (Line 9), and then chooses the action based on a dynamically updated throttling parameter (Line 10). The throttling parameter λ_t is updated afterward according to the estimated cost to control the rate of budget expenditure (Line 12). Intuitively, a large λ_t implies that the budget is spent too fast currently, and the algorithm reduces the frequency of entering the market. Conversely, an average expenditure below the ideal ρ in past rounds will result in a descent of λ_t and encourage the algorithm to participate in the auction.

The choice of x_t and the update rule of λ_t aim at approximating the best solution of programming (1) with hindsight in the dual space. Let the expected revenue and cost given v be $r(v) := \mathbb{E}_{p\sim G} [(v-p)^+]$ and $c(v) := \mathbb{E}_{p\sim G} [p\mathbf{1}[v \ge p]]$, then the dual problem of programming (1) is

$$\min_{\lambda \ge 0} \max_{\pi: [0,1] \mapsto [0,1]} T \cdot \left(\mathbb{E}_{v} \left[\pi(v) \cdot \left(r(v) - \lambda c(v) \right) \right] + \lambda \rho \right), \tag{2}$$

where λ is the Lagrangian multiplier. For a fixed λ , the objective is maximized by $\pi(v) = \mathbf{1}[r(v) \ge \lambda c(v)]$, i.e. choosing to enter for all auctions with positive $r(v) - \lambda c(v)$. Thus, we obtain an upper bound of OPT by weak duality,

$$OPT \le \inf_{\lambda \ge 0} T \cdot \left(\mathbb{E}_v \left[(r(v) - \lambda c(v))^+ \right] + \lambda \rho \right).$$
(3)

Since the buyer cannot compute r(v), c(v) and λ^* without prior information on the real distributions F and G, the proposed throttling strategy chooses the action in round t based on the estimates $\tilde{r}_t(v_t), \tilde{c}_t(v_t)$ and λ_t instead.

3.2 Full Information Feedback Setting

In the full information setting, the buyer can always observe the price p_t in round t whatever her action is. We first that in this scenario, OGD-CB achieves an $O(\sqrt{T \log T})$ regret bound against OPT.

Theorem 3.1. With full information feedback, there is a constant C^{F} , such that with probability at least 1 - 4/T, Algorithm 1 guarantees

$$Reg^{\beta}(\boldsymbol{v},\boldsymbol{p}) \leq C^{\mathrm{F}}\sqrt{T\ln T}.$$

Input: ρ, T . **Initialization:** $\mathcal{I}_1 \leftarrow \emptyset, B_1 \leftarrow B, \lambda_1 \leftarrow 0.$ 1 for $t \leftarrow 1$ to T do $\mathbf{2}$ Observe v_t ; /* A single round of exploration. */ if t = 1 then 3 Choose $x_t \leftarrow 1$; $\mathbf{4}$ Update $\lambda_{t+1} = \lambda_t$; 5 end 6 else $\mathbf{7}$ /* Estimate the revenue and cost with confidence bound. */ $\begin{aligned} &\epsilon_t \leftarrow \sqrt{(\ln 2 + 2\ln T)/(2|\mathcal{I}_t|)};\\ &\widetilde{r}_t(v_t) \leftarrow \frac{1}{|\mathcal{I}_t|} \sum_{\tau \in \mathcal{I}_t} (v_t - p_\tau)^+ + \epsilon_t v_t, \ \widetilde{c}_t(v_t) \leftarrow \frac{1}{|\mathcal{I}_t|} \sum_{\tau \in \mathcal{I}_t} p_\tau \mathbf{1}[v_t \ge p_\tau] - 2\epsilon_t v_t; \end{aligned}$ 8 9 /* Choose the action according to the estimates. */ $x_t \leftarrow \mathbf{1}[\widetilde{r}_t(v_t) \ge \lambda_t \widetilde{c}_t(v_t)];$ $\mathbf{10}$ /* Online gradient descent on the dual variable. */ $\eta_t \leftarrow (1/\rho - 1)/\sqrt{t};$ 11 $\lambda_{t+1} \leftarrow (\lambda_t + \eta_t (x_t \widetilde{c}_t (v_t) - \rho))^+;$ 1213 end /* Observe the sample. */ if $(FULL-INFO) \lor (PARTIAL-INFO \land x_t = 1)$ then $\mathbf{14}$ Observe p_t ; $\mathbf{15}$ $\mathcal{I}_{t+1} \leftarrow \mathcal{I}_t \cup \{t\};$ 16end $\mathbf{17}$ else 18 $\mathcal{I}_{t+1} \leftarrow \mathcal{I}_t;$ 19 end $\mathbf{20}$ /* Update the remaining budget. */ $B_{t+1} \leftarrow B_t - x_t c_t;$ $\mathbf{21}$ if $B_{t+1} < 1$ then $\mathbf{22}$ break; $\mathbf{23}$ end $\mathbf{24}$ 25 end

To prove the theorem, we first need some concentration bounds for the estimation process in OGD-CB. Below we show that $\tilde{r}_t(v_t)$ and $\tilde{c}_t(v_t)$ are good estimations of $r(v_t) := \mathbb{E}_{p\sim G}[(v_t - p)^+]$ and $c(v_t) := \mathbb{E}_{p\sim G}[p\mathbf{1}[v_t \ge p]]$ correspondingly, by applying Dvoretzky–Kiefer–Wolfowitz (DKW) inequality [45].

Lemma 3.2. With full information feedback, in round $1 < t \leq T_0$, with failure probability $q(t, \epsilon_t) := 2 \exp(-2(t-1)\epsilon_t^2)$, for any $v_t \in [0, 1]$, we have

$$r(v_t) \le \widetilde{r}_t(v_t) \le r(v_t) + 2\epsilon_t v_t, \tag{4}$$

$$c(v_t) - 4\epsilon_t v_t \le \tilde{c}_t(v_t) \le c(v_t).$$
(5)

Another crucial ingredient of the proof concerns with the optimization process in Line 12 of OGD-CB. Lemma 3.3 gives a useful observation about λ_t .

Lemma 3.3. For any $t \leq T_0 + 1$, we have $\lambda_t \in [0, 1/\rho - 1]$.

With Lemma 3.3 in hand, a standard analysis of the online gradient descent method [36] with online gain function $h_t(\lambda) = \lambda(x_t \tilde{c}_t(v_t) - \rho)$ gives the following result:

Lemma 3.4. For any $\lambda \in [0, 1/\rho - 1]$, we have

$$\sum_{t=2}^{T_0} \lambda_t (x_t \widetilde{c}_t(v_t) - \rho) \ge \lambda \sum_{t=2}^{T_0} (x_t \widetilde{c}_t(v_t) - \rho) - \left(\frac{(1/\rho - 1)^2}{\eta_{T_0}} + \sum_{t=2}^{T_0} \eta_t\right).$$
(6)

Here, we provide a high-level proof sketch of Theorem 3.1.

Proof Sketch of Theorem 3.1. We prove the result in four steps.

Step 1: We first lower bound the performance of our throttling strategy by using Azuma-Hoeffding inequality and Lemma 3.2. We arrive at

$$U^{\beta}(\boldsymbol{v},\boldsymbol{p}) \geq \sum_{t=2}^{T_0} x_t \widetilde{r}_t(v_t) - O\left(\sqrt{T\log T}\right).$$

Step 2: We next bound $\sum_{t=2}^{T_0} x_t \widetilde{r}_t(v_t)$ by rewriting it as

$$\sum_{t=2}^{T_0} x_t \widetilde{r}_t(v_t) = \underbrace{\sum_{t=2}^{T_0} \left(x_t \widetilde{r}_t(v_t) - \lambda_t \left(x_t \widetilde{c}_t(v_t) - \rho \right) \right)}_{R_1} + \underbrace{\sum_{t=2}^{T_0} \lambda_t \left(x_t \widetilde{c}_t(v_t) - \rho \right)}_{R_2},$$

and bounding R_1, R_2 respectively. It is worth mentioning that R_1 looks quite similar to the objective of the dual problem in (2), which intuitively implies R_1 is "close" to $(T_0 - 1)\text{OPT}/T$. By applying Azuma-Hoeffding inequality and Lemma 3.2 again, we obtain

$$R_1 \ge \frac{T_0 - 1}{T} \operatorname{OPT} - O\left(\sqrt{T \log T}\right).$$

As for R_2 , we apply Lemma 3.4 to obtain

$$R_2 \ge -O\left(\sqrt{T}\right).$$

Step 3: The third step proves that the ending time T_0 of our strategy is close to T. More precisely, we show that

$$\frac{T_0 - 1}{T} \text{OPT} \ge \text{OPT} - O\left(\sqrt{T \log T}\right).$$

Step 4: Finally, the proof is completed by putting everything together and using a union bound to lower bound the total failure probability. □

3.3 Partial Information Feedback Setting

In the partial information setting, the buyer can observe p_t only if she enters the round t. We show that the upper bound of $O(\sqrt{T \log T})$ on the regret still holds for this information structure.

Theorem 3.5. With partial information feedback, there is a constant C^{P} , such that with probability at least 1 - 4/T, Algorithm 1 guarantees

$$Reg^{\beta}(\boldsymbol{v},\boldsymbol{p}) \leq C^{\mathrm{P}}\sqrt{T\ln T}$$

As a preparation for the proof of Theorem 3.5, we first give an analog of Lemma 3.2 in the partial information feedback setting without proof.

Lemma 3.6. With partial information feedback, in round $t \ge 1$ with failure probability $q'(t, \epsilon_t) := 2 \exp(-2|\mathcal{I}_t|\epsilon_t^2)$, for any $v_t \in [0, 1]$, we have

$$r(v_t) \le \widetilde{r}_t(v_t) \le r(v_t) + 2\epsilon_t v_t,$$

$$c(v_t) - 4\epsilon_t v_t \le \widetilde{c}_t(v_t) \le c(v_t).$$

The main challenge of regret analysis with partial information is that the algorithm may not collect enough historical data to give good estimates of $r(v_t)$ and $c(v_t)$. In other words, we may not be able to bound the failure probability and the estimation error at the same time. We overcome this challenge by bounding the entering frequency. Specifically, we use an induction method to show that $|\mathcal{I}_t|$, the entering frequency before round t, grows with t in an approximately linearly.

Lemma 3.7. Let $C_e := \min\{(\sqrt{2}/2)\rho^2, (\sqrt{2}/4)\rho\}$. In the partial information setting, for any $t \ge 2$, Algorithm 1 guarantees that $|\mathcal{I}_t| \ge C_e \cdot (t-1)$.

With the help of Lemma 3.6 and Lemma 3.7, we can bound the accumulated inaccuracy in estimating $r(v_t)$ and $c(v_t)$ with partial information feedback. The remaining proof of Theorem 3.5 is almost identical to the proof of Theorem 3.1.

4 Lower Bound on the Regret of Online Throttling

This section shows that for any online throttling strategy, its expected regret facing OPT is $\Omega(\sqrt{T})$ even in the full information feedback setting. Our main result in this section is the following theorem:

Theorem 4.1. There exists an instance tuple (F, G, ρ) , such that there is a constant $C_l > 0$, for any online throttling strategy β and 4|T, we have

$$\mathbb{E}_{\boldsymbol{v} \sim F^T, \boldsymbol{p} \sim G^T} \left[Reg^{\beta}(\boldsymbol{v}, \boldsymbol{p}) \right] \geq C_l \sqrt{T}.$$

Similar results have already existed in [9,21,49] in the literature of dynamic resource-constrained reward collection problems [12]. Nevertheless, these works consider the difference between hindsight and fluid bi-directionally adaptive benchmarks. (See Appendix A.) Here, we give a proof exclusive to our setting. The central idea lying in the proof is to give a proper problem instance, under which even the optimal throttling strategy with the hindsight of \boldsymbol{v} and \boldsymbol{p} does not escape a regret of $\Omega(T)$.

Specifically, we consider the problem instance with v always equal to 1, G be the uniform binary distribution on 1/3 and 2/3, and $\rho = 1/2$. We first establish an upper bound on $U^{\beta}(v, p)$

for any online throttling strategy β when 4|T. Then we give a non-simplified formula for the lower bound on regret in terms of OPT. Finally, we simplify this lower bound and use Stirling's formula. Readers can refer to a full proof in Appendix C.

Note that the performance of an online throttling strategy could only be worse with partial information. Therefore, Theorem 4.1 also implies that no online throttling strategy can obtain a better regret in hindsight with partial information feedback. Together with Theorem 3.1 and Theorem 3.5, this establishes the near-optimality of our OGD-CB algorithm in both information settings.

5 Comparison between Online Throttling and Online Pacing

In this section, we focus on comparing two popular online budget control methods. The first one is online throttling, which is studied in this work. The second is online pacing [14], which is extensively studied in literature and widely adopted in the industry. Specifically, we are most interested in the comparison of the two strategies. Nevertheless, it is hard to compare two adaptive learning procedures theoretically. Therefore, we consider the problem in the fluid sense. Also, our discussion extends the result in [15], which compares these two strategies in system equilibrium.

To start with, we define the fluid benchmark given by the optimal pacing strategy as:

$$OPT_{P} := T \cdot \max_{\mu \ge 0} \mathbb{E}_{v \sim F, p \sim G} \left[(v - p) \mathbf{1} [v \ge (1 + \mu)p] \right], \quad \text{s.t.} \quad \mathbb{E}_{v \sim F, p \sim G} \left[p \mathbf{1} [v \ge (1 + \mu)p] \right] \le \rho.$$

 OPT_P captures the optimal expected revenue brought by any pacing strategy not exceeding the budget in expectation.

We now focus on comparing OPT and OPT_P . Our answer to the problem is two-fold: (1) In the very general sense, the numerical order of the revenue brought by these two strategies can be arbitrary. (2) When the optimal pacing strategy exhausts the budget, then this strategy is even better than the optimal adaptive throttling strategy.

In the general case, we have the following lemma, indicating that optimal throttling and pacing can have arbitrary numerical order.

Lemma 5.1. In different cases, any of $OPT > OPT_P$, $OPT = OPT_P$ and $OPT < OPT_P$ may happen.

We now prove Lemma 5.1 with an example.

Example 5.1. We now discuss the scenario in which v follows a uniform binary distribution on 3/8 and 1, while p follows a uniform binary distribution on 1/4 and 3/4. Meanwhile, we assume T = 1. We analyze the numerical relationships of the three benchmarks when ρ grows from 1/8 to 1/4.

- For optimal pacing, we have $\mu^* = 1/2$, and $OPT_P = 7/32$ for any value ρ in [1/8, 1/4].
- For optimal adaptive throttling, we always have $\pi^*(3/8) = 0$. Meanwhile, $\pi^*(1)$ grows linearly from 1/2 to 1 with ρ growing from 1/8 to 1/4, and OPT grows correspondingly from 1/8 to 1/4.

Since 1/4 > 7/32 > 1/8, by continuity, any of the three events may occur with different $\rho \in [1/8, 1/4]$.

Nevertheless, when the optimal pacing strategy accurately exhausts the budget in expectation, we can show that it is no worse strategy than optimal adaptive/fixed throttling. In fact, from the proof of the following lemma, we can even demonstrate that it is the best strategy in the expected sense.

Lemma 5.2. If there is some $\mu^* \geq 0$ such that $\mathbb{E}_{v \sim F, p \sim G}[p\mathbf{1}[v \geq (1 + \mu^*)p]] = \rho$, then $OPT_P \geq OPT$ holds.

Proof of Lemma 5.2. For any $v, p \in [0, 1]$, we let $J(v, p) \in [0, 1]$ be a generalized choice function which indicates the probability that the buyer agrees to enter the auction in with knowledge of her value v and the highest competitive bid p. We consider the following optimization problem, which gives the fluid bi-directionally adaptive benchmark (Appendix A):

$$OPT_{S} := T \cdot \max_{J:[0,1]^2 \to [0,1]} \mathbb{E}_{v \sim F, p \sim G} \left[(v-p)J(v,p) \right], \quad \text{s.t.} \quad \mathbb{E}_{v \sim F, p \sim G} \left[pJ(v,p) \right] \le \rho.$$

Clearly, by definition, $OPT \leq OPT_S$ and $OPT_P \leq OPT_S$ hold. By duality, we have

$$OPT_{S} \leq \min_{\mu \geq 0} \max_{J:[0,1]^{2} \to [0,1]} T \cdot \mathbb{E}_{v \sim F, p \sim G} \left[(v - (1+\mu)p)J(v,p) \right] + \mu\rho T$$
$$= \min_{\mu \geq 0} T \cdot \mathbb{E}_{v \sim F, p \sim G} \left[(v - (1+\mu)p)^{+} \right] + \mu\rho T.$$

Here, μ is the dual variable. Now, let $\mu^* \geq 0$ satisfy $\mathbb{E}_{v \sim F, p \sim G}[p\mathbf{1}[v \geq (1 + \mu^*)p]] = \rho$. As a result, we have

$$\begin{aligned}
\text{OPT}_{P} &\geq T \cdot \mathbb{E}_{v \sim F, p \sim G} \left[(v - p) \mathbf{1} [v \geq (1 + \mu^{*}) p] \right] \\
&= T \cdot \mathbb{E}_{v \sim F, p \sim G} \left[(v - p) \mathbf{1} [v \geq (1 + \mu^{*}) p] \right] - \mu^{*} T (\mathbb{E}_{v \sim F, p \sim G} \left[p \mathbf{1} [v \geq (1 + \mu^{*}) p] \right] - \rho) \\
&= \mathbb{E}_{v \sim F, p \sim G} \left[(v - (1 + \mu^{*}) p)^{+} \right] + \mu^{*} \rho T \\
&\geq \text{OPT},
\end{aligned}$$

which indicates that $OPT_P = OPT_S \ge OPT$.

In Appendix A, we add three more benchmarks into the comparison, hoping to give a panorama of these two budget control methods and the optimal benchmarks in the dynamic setting. Despite the difference in performance, we want further to emphasize the different application scenarios of these two strategies. A significant distinction between throttling and pacing is that the buyer can modify the bid under pacing, but cannot with throttling. In real-life, with sufficient market insight, giant advertisers tend to hold the bidding rights in their hands. However, such an option may not be applicable for small advertisers, since they do not have enough knowledge and prediction on the market. Consequently, bidding by themselves may lead to inappropriate usage of the budget and a loss of revenue. On the other hand, it might be a better choice for them to get a recommended bid from the platform, which has a complete view of the market. The above is possibly the main reason why the market embraces both budget management strategies.

6 Concluding Remarks

As a problem that originated from the practical side, dynamic throttling in second-price auction finds its place in the wide range of contextual bandits problems with knapsacks. Nonetheless, this problem is substantially different from other closely related problems in the sense that the distributions of the reward and cost are both unknown to the deciding agent. Such a difference turns out to be a determining factor for the regret bound of the algorithm compared to other problems. Consequently, a natural question is that whether the $O(\sqrt{T \log T})$ upper bound and the $\Omega(\sqrt{T})$ lower bound can be closed. Meanwhile, we hope for a more thorough understanding of the fluid throttling/pacing benchmarks and the hindsight benchmark (Appendix A), especially their approximation relationships.

Another possible direction exclusive to our problem is strengthening the information structure so that the buyer can only see the highest competing bid when she wins the auction. Clearly, under such a setting, the samples seen by the buyer are biased. It is interesting to see whether we can simultaneously conduct the learning and optimizing process in this setting.

A Reasoning on Different Benchmarks

In this section, we give a synthesized analysis of different benchmarks for budget control methods in repeated second-price auctions. Specifically, we consider four fluid benchmarks and the hindsight benchmark OPT_H . These four fluid benchmarks include the fixed-throttling benchmark OPT_0 , the adaptive-throttling benchmark which we mainly focus on in this paper OPT, the pacing benchmark OPT_P and the bidirectionally adaptive benchmark OPT_S which we come through in the proof of Lemma 5.2.

A.1 The Fluid Fixed-Throttling Benchmark

To start this section, we first consider the fluid fixed-throttling benchmark OPT_0 , which corresponds to the optimal *fixed* throttling strategy for all possible values. Concretely, it has the following definition:

$$OPT_0 := T \cdot \max_{\theta \in [0,1]} \mathbb{E}_{v \sim F, p \sim G} \left[\theta \cdot (v-p)^+ \right], \quad \text{s.t.} \quad \mathbb{E}_{v \sim F, p \sim G} \left[\theta \cdot p \mathbf{1}[v \ge p] \right] \le \rho.$$
(7)

Clearly, the feasible set of θ in programming (7) is a subset of the feasible set in programming (1). Therefore, it is certain that $OPT_0 \leq OPT$. In fact, it is common that $OPT - OPT_0 = \Theta(T)$. Example A.1 gives an instance.

We now show that OPT_0 is weak in the sense that even the naïve online algorithm that always enters can guarantee an $O(\sqrt{T \log T})$ regret over OPT_0 .

Lemma A.1. Let \mathcal{N} be the strategy that always enters until the budget is depleted. Then, with probability 1 - O(1/T),

$$OPT_0 - U^{\mathcal{N}}(\boldsymbol{v}, \boldsymbol{p}) = O(\sqrt{T \log T}).$$

Proof of Lemma A.1. Let $R := \mathbb{E}_{v \sim F, p \sim G}[(v-p)^+]$ and $C := \mathbb{E}_{v \sim F, p \sim G}[p\mathbf{1}[v \geq p]]$. We immediately have $OPT_0 = \rho RT/C$. By Hoeffding's inequality, we have with probability 1 - 2/T,

$$\sum_{t=1}^{T_0} r_t \ge RT_0 - \sqrt{\frac{T \ln T}{2}}, \quad \rho T \le \sum_{t=1}^{T_0} c_t + 1 \le CT_0 + \sqrt{\frac{T \ln T}{2}} + 1.$$

Combining the two inequalities, we derive the lemma.

With the fixed-throttling benchmark OPT_0 , we can give a more thorough comparison between the throttling strategy and the pacing strategy in the fluid sense, which is an extension of the result in Section 5. An interesting conclusion is that even OPT_0 , which is always no larger than OPT, could be larger than OPT_P in certain circumstances.

Lemma A.2 (An extension of Lemma 5.1). In different cases, any of $OPT_P > OPT > OPT_0$, $OPT > OPT_P > OPT_0$ and $OPT > OPT_0 > OPT_P$ may happen.

Similarly, the proof is done by considering OPT_0 in Example 5.1.

Example A.1. We now consider OPT_0 in the setting of Example 5.1. Specifically, when ρ increases from 1/8 to 1/4, θ^* increases from 2/5 to 4/5 linearly with ρ , and OPT_0 respectively increases from 9/80 to 9/40. Since 1/4 > 9/40 > 7/32 > 1/8 > 9/80, any of the three events may occur with different $\rho \in [1/8, 1/4]$.

A.3 Towards the Optimal Revenue

Now we come to define two benchmarks which characterize the optimal revenue of any bidding mechanisms. These two benchmarks are widely studied in literature other than auction. The first is hindsight benchmark (a.k.a. hindsight optimum [21]), with the following definition:

$$OPT_{\mathrm{H}} := \mathbb{E}_{\boldsymbol{v} \sim F^{T}, \boldsymbol{p} \sim G^{T}} \left[U_{\mathrm{H}}(\boldsymbol{v}, \boldsymbol{p}) \right]$$
$$U_{\mathrm{H}}(\boldsymbol{v}, \boldsymbol{p}) := \max_{\boldsymbol{x} \in \{0,1\}^{T}} \sum_{t=1}^{T} x_{t} (v_{t} - p_{t})^{+}, \quad \text{s.t.} \sum_{t=1}^{T} x_{t} p_{t} \mathbf{1} [v_{t} \ge p_{t}] \le \rho T.$$

The second optimum, which we refer to as the fluid bi-directionally adaptive benchmark (a.k.a. fluid benchmark [49] or deterministic LP [21] in the discrete case) is called as the fluid benchmark or deterministic LP in literature, has already appeared in the proof of Lemma 5.2. We explicitly give the definition again for clearness:

$$OPT_{S} := T \cdot \max_{J:[0,1]^2 \to [0,1]} \mathbb{E}_{v \sim F, p \sim G} \left[(v-p)J(v,p) \right], \quad \text{s.t.} \quad \mathbb{E}_{v \sim F, p \sim G} \left[pJ(v,p) \right] \le \rho.$$

As a direct consequence of the definition, we have $OPT_S \ge OPT \ge OPT_0$ and $OPT_S \ge OPT_P$. Meanwhile, we derive the following result from the proof of Lemma 5.2.

Lemma A.3. If there is some $\mu^* \geq 0$ such that $\mathbb{E}_{v \sim F, p \sim G}[p\mathbf{1}[v \geq (1 + \mu^*)p]] = \rho$, then $OPT_P = OPT_S$.

We are much interested in how OPT_H compares with these four benchmarks. To start with, we review some results in literature on comparing OPT_S and OPT_H . The most fundamental and well-known result tells that $OPT_S \ge OPT_H$ by Jensen's inequality. Further, we have the following two results from previous works.

Proposition A.4 (From [9,21,49]). There exists an instance tuple (F, G, ρ) , under which $OPT_S - OPT_H = \Theta(\sqrt{T})$ holds.

Proposition A.5 (From [37, 50]). There exists an instance tuple (F, G, ρ) , under which $OPT_S - OPT_H = \Theta(1)$ holds.

We extend the above results by getting the other three benchmarks involved. To start with, we notice that in the example for the lower bound in Theorem 4.1, all four benchmarks OPT_S , OPT_O , OPT_O , and OPT_P coincide. Therefore, we have the following lemma.

Lemma A.6. There exists an instance tuple (F, G, ρ) , under which $OPT_S = OPT_P = OPT = OPT_0$ and $OPT_S - OPT_H = \Theta(\sqrt{T})$ simultaneously hold.

Further, we give another instance, which illustrates the following lemma.

Lemma A.7. There exists an instance tuple (F, G, ρ) , under which $OPT_P = OPT = OPT_0$, $OPT_S - OPT_H = \Theta(1)$, and $OPT_S - OPT = \Theta(T)$ simultaneously hold.

Proof of Lemma A.7. For the instance, we let v always equal to 1, G be the uniform binary distribution on 1/3 and 2/3, and $\rho = 1/3$. Therefore, in this instance,

- $J^*(1, 1/3) = 1$, $J^*(1, 2/3) = 1/2$, and $OPT_S = (5/12)T$;
- $\mu^* = 2$, and $OPT_P = (1/3)T$;

•
$$\pi^*(1) = \theta^* = 2/3$$
, and OPT = OPT₀ = $(1/3)T$.

We now calculate OPT_H when 2|T. We let S be the number of 1/3(s) in p, then by a similar argument in the proof of Theorem 4.1, we have

$$U_{\mathrm{H}}(\boldsymbol{v},\boldsymbol{p}) = rac{2}{3} \cdot S + rac{1}{3} \cdot \left\lfloor rac{T-S}{2}
ight
floor.$$

Therefore, we derive that

$$OPT_{H} = \frac{1}{2^{T}} \sum_{S=0}^{T} \left(\frac{2}{3} \cdot S + \frac{1}{3} \cdot \left\lfloor \frac{T-S}{2} \right\rfloor \right) \begin{pmatrix} T\\ S \end{pmatrix}$$
$$= \frac{1}{2^{T}} \sum_{S=0}^{T} \left(\frac{1}{6}T + \frac{1}{2}S \right) \begin{pmatrix} T\\ S \end{pmatrix} - \frac{1}{6 \cdot 2^{T}} \sum_{t=1}^{T/2} \begin{pmatrix} T\\ 2t-1 \end{pmatrix}$$
$$\stackrel{(a)}{=} \frac{1}{6}T + \frac{1}{2^{T}} \sum_{S=0}^{T-1} \begin{pmatrix} T-1\\ S \end{pmatrix} - \frac{1}{12} = \frac{5}{12}T - \frac{1}{12}.$$

Here, (a) is because $S \cdot {T \choose S} = T \cdot {T-1 \choose S-1}$ for $1 \le S \le T$ and $\sum_{t=1}^{T/2} {T \choose 2t-1} = 2^{T-1}$. Hence the lemma is proved.

B Missing Proofs in Section **3**

B.1 Proof of Lemma 3.2

In the full-information setting, the buyer can observe all the history prices (Line 14), i.e. for any t > 1, $|\mathcal{I}_t| = \{1, 2, \dots, t-1\}$. Note that Line 9 of OGD-CB essentially estimates $r(v_t)$ and $c(v_t)$ according to an empirical distribution \hat{G}_t . In detail, we let \hat{G}_t be the distribution of (t-1)independent samples of p, then we have

$$\widetilde{r}_{t}(v_{t}) = \frac{1}{t-1} \sum_{1 \le \tau \le t-1} (v_{t} - p_{\tau})^{+} + \epsilon_{t} v_{t} = \mathbb{E}_{p \sim \widehat{G}_{t}} \left[(v_{t} - p)^{+} \right] + \epsilon_{t} v_{t},$$

$$\widetilde{c}_{t}(v_{t}) = \frac{1}{t-1} \sum_{1 \le \tau \le t-1} p_{\tau} \mathbf{1} [v_{t} \ge p_{\tau}] - 2\epsilon_{t} v_{t} = \mathbb{E}_{p \sim \widehat{G}_{t}} \left[p \mathbf{1} [v_{t} \ge p] \right] - 2\epsilon_{t} v_{t}.$$

We now do a calculation on $r(v_t)$ and $c(v_t)$. By using integration by parts, we have

$$r(v_t) = \mathbb{E}_{p \sim G} \left[(v_t - p)^+ \right] = \int_0^{v_t} (v_t - p) \, \mathrm{d}G(p) = -\int_0^{v_t} G(p) \, \mathrm{d}(v_t - p) = \int_0^{v_t} G(p) \, \mathrm{d}p,$$

$$c(v_t) = \mathbb{E}_{p \sim G} \left[p \mathbf{1}[v_t \ge p] \right] = \int_0^{v_t} pg(p) \, \mathrm{d}p = v_t G(v_t) - \int_0^{v_t} G(p) \, \mathrm{d}p.$$

By DKW inequality, for any $\epsilon_t \geq 0$, we have

$$\Pr\left|\sup_{p\in[0,1]} \left|\widehat{G}_t(p) - G(p)\right| \ge \epsilon_t\right| \le 2\exp\left(-2(t-1)\epsilon_t^2\right) = q(t,\epsilon_t).$$

Suppose that $\left|\widehat{G}_t(p) - G(p)\right| \leq \epsilon_t$ happens for every $p \in [0, 1]$, then for any $v_t \in [0, 1]$, we have

$$\left|\mathbb{E}_{p\sim\widehat{G}_t}\left[(v_t-p)^+\right] - \mathbb{E}_{p\sim G}\left[(v_t-p)^+\right]\right| = \left|\int_0^{v_t} \left(\widehat{G}_t(p) - G(p)\right) \,\mathrm{d}p\right| \le \epsilon_t v_t,$$

$$\left|\mathbb{E}_{p\sim\widehat{G}_t}\left[p\mathbf{1}[v_t\geq p]\right] - \mathbb{E}_{p\sim G}\left[p\mathbf{1}[v_t\geq p]\right]\right| = \left|v_t\left(\widehat{G}_t(v_t) - G(v_t)\right) - \int_0^{v_t}\left(\widehat{G}_t(p) - G(p)\right) \,\mathrm{d}p\right| \le 2\epsilon_t v_t.$$

As a result, by the definition of $\tilde{r}_t(\cdot)$ and $\tilde{c}_t(\cdot)$ in Algorithm 1, we derive the lemma.

B.2 Proof of Lemma 3.3

First, observe that

$$\widetilde{r}_t(v_t) = \mathbb{E}_{p \sim \widehat{G}_t} \left[(v_t - p)^+ \right] + \epsilon_t v_t = \mathbb{E}_{p \sim \widehat{G}_t} \left[v_t \mathbf{1} \left[v_t \ge p_t \right] \right] - \mathbb{E}_{p \sim \widehat{G}_t} \left[p_t \mathbf{1} \left[v_t \ge p_t \right] \right] + \epsilon_t v_t = \mathbb{E}_{p \sim \widehat{G}_t} \left[v_t \mathbf{1} \left[v_t \ge p_t \right] \right] - \widetilde{c}_t(v_t) - \epsilon_t v_t.$$

Then by the choice of x_t , we have

$$x_t \tilde{c}_t(v_t) \le \frac{x_t}{1+\lambda_t} \left(\mathbb{E}_{p_t \sim \widehat{G}_t} \left[v_t \mathbf{1} \left[v_t \ge p_t \right] \right] - \epsilon_t v_t \right) \le \frac{1}{1+\lambda_t},\tag{8}$$

since the value of $x_t(\tilde{r}_t(v_t) - \lambda_t \tilde{c}_t(v_t))$ must be no less than 0. If we assume $\lambda_t \leq \frac{1}{\rho} - 1$, we have

$$\lambda_{t+1} = (\lambda_t + \eta_t (x_t \tilde{c}_t (v_t) - \rho))^+ \stackrel{(a)}{\leq} \left(\lambda_t + \frac{\eta_t}{1 + \lambda_t} - \eta_t \rho\right)^+$$
$$\stackrel{(b)}{\leq} \max\left\{\eta_t (1 - \rho), \frac{1}{\rho} - 1, 0\right\} \stackrel{(c)}{=} \frac{1}{\rho} - 1.$$

In the derivation above, (a) follows the bound of $x_t \tilde{c}_t(v_t)$ in (8); (b) is due to the convexity of $\lambda_t + \frac{\eta_t}{1+\lambda_t}$ and $\lambda_t \in [0, 1/\rho - 1]$; (c) uses the fact that $\eta_t \leq 1$ and $\rho \leq 1$. Since $\lambda_1 = \lambda_2 = 0 \leq \frac{1}{\rho} - 1$, by induction we obtain that $\lambda_t \leq \frac{1}{\rho} - 1$ uniformly for all $t \leq T_0 + 1$.

B.3 Proof of Lemma 3.4

By the update rule of λ_t , we have for any $\lambda \in [0, 1/\rho - 1]$,

$$\|\lambda_{t+1} - \lambda\|^2 \le \|\lambda_t + \eta_t (x_t \widetilde{c}_t(v_t) - \rho) - \lambda\|^2$$

$$\frac{1}{\eta_t} \left(\|\lambda_{t+1} - \lambda\|^2 - \|\lambda_t - \lambda\|^2 \right) \le (\lambda_t - \lambda) (x_t \widetilde{c}_t(v_t) - \rho) + \eta_t \| (x_t \widetilde{c}_t(v_t) - \rho) \|^2.$$

A telescoping summation from t = 2 through T_0 gives

$$\begin{split} \sum_{t=2}^{T_0} (\lambda_t - \lambda) (x_t \tilde{c}_t(v_t) - \rho) &\geq \sum_{t=2}^{T_0} \frac{1}{\eta_t} \left(\|\lambda_{t+1} - \lambda\|^2 - \|\lambda_t - \lambda\|^2 \right) - \sum_{t=2}^{T_0} \eta_t \| (x_t \tilde{c}_t(v_t) - \rho\|^2 \\ &\stackrel{(a)}{\geq} \sum_{t=2}^{T_0} \frac{1}{\eta_t} \left(\|\lambda_{t+1} - \lambda\|^2 - \|\lambda_t - \lambda\|^2 \right) - \sum_{t=2}^{T_0} \eta_t \\ &\stackrel{(b)}{\geq} - \left(\frac{1}{\rho} - 1 \right)^2 \sum_{t=3}^{T_0} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) - \sum_{t=2}^{T_0} \eta_t \\ &\geq - \frac{(1/\rho - 1)^2}{\eta_{T_0}} - \sum_{t=2}^{T_0} \eta_t, \end{split}$$

where (a) holds since $|(x_t \tilde{c}_t(v_t) - \rho| \le 1 \text{ and (b)}$ follows from Lemma 3.3 and that η_t decreases in t.

B.4 Proof of Theorem 3.1

Throughout the proof, we assume that $\rho \leq 1$, otherwise the problem is trivial. We prove the result in four steps. We first lower bound the performance of our throttling strategy by using Azuma-Hoeffding inequality and Lemma 3.2. Then we rewrite the lower bound into two parts and bound each part respectively. The third step proves that the ending time T_0 of our strategy is close to T. Finally, we conclude by combining these steps to obtain the desired result.

Step 1: lower bound on the performance. We start with lower-bounding the total reward that the algorithm gets. Notice that

$$\mathbb{E}_{v_t \sim F, p_t \sim G} \left[x_t (r_t - r(v_t)) \mid \mathcal{H}_t \right] = 0$$

Thus, by applying Azuma–Hoeffding inequality, we have

$$U^{\beta}(\boldsymbol{v}, \boldsymbol{p}) = \sum_{t=1}^{T_0} x_t r_t \ge \sum_{t=1}^{T_0} x_t r(v_t) - \sqrt{2T \ln T},$$
(9)

with a failure probability of 1/T. Suppose that the event stated in Lemma 3.2 happens, i.e. all the estimations are successful, which holds with probability at least $1 - \sum_{t=2}^{T_0} q(t, \epsilon_t)$. In this case, (4) establishes for all $2 \le t \le T_0$ and hence

$$\sum_{t=2}^{T_0} x_t r(v_t) \ge \sum_{t=2}^{T_0} x_t \widetilde{r}_t(v_t) - 2 \sum_{t=2}^{T_0} x_t \epsilon_t v_t.$$
(10)

Here, we start the summation from t = 2 since $\tilde{r}_t(\cdot)$ is not defined for t = 1. By the construction of $\epsilon_t = \sqrt{(\ln 2 + 2 \ln T)/(2(t-1))}$ with full-information feedback, we have

$$\sum_{t=2}^{T_0} \epsilon_t \le \sqrt{2T(\ln 2 + 2\ln T)} \text{ , and } \sum_{t=2}^{T_0} q(t, \epsilon_t) \le 1/T.$$
(11)

Combining equations (9), (10) and (11), with probability at least 1 - 2/T, we have

$$U^{\beta}(\boldsymbol{v}, \boldsymbol{p}) \ge \sum_{t=2}^{T_0} x_t \tilde{r}_t(v_t) - \sqrt{2T \ln T} - 2\sqrt{2T(\ln 2 + 2\ln T)}.$$
 (12)

Step 2: bounding $\sum_{t=2}^{T_0} x_t \tilde{r}_t(v_t)$. We next consider the first term in the right hand side of (12). Rewriting it into two parts, we have

$$\sum_{t=2}^{T_0} x_t \widetilde{r}_t(v_t) = \underbrace{\sum_{t=2}^{T_0} (x_t \widetilde{r}_t(v_t) - \lambda_t (x_t \widetilde{c}_t(v_t) - \rho))}_{R_1} + \underbrace{\sum_{t=2}^{T_0} \lambda_t (x_t \widetilde{c}_t(v_t) - \rho)}_{R_2}.$$
 (13)

Let $\pi^* : [0,1] \mapsto [0,1]$ be the optimal solution for the benchmark OPT defined in Program (1). We have

$$R_{1} = \sum_{t=2}^{T_{0}} x_{t}(\tilde{r}_{t}(v_{t}) - \lambda_{t}\tilde{c}_{t}(v_{t})) + \sum_{t=2}^{T_{0}} \lambda_{t}\rho \stackrel{(a)}{\geq} \sum_{t=2}^{T_{0}} \pi^{*}(v_{t})(\tilde{r}_{t}(v_{t}) - \lambda_{t}\tilde{c}_{t}(v_{t})) + \sum_{t=2}^{T_{0}} \lambda_{t}\rho$$

$$\stackrel{\text{(b)}}{\geq} \sum_{t=2}^{T_0} \pi^*(v_t)(r(v_t) - \lambda_t c(v_t)) + \sum_{t=2}^{T_0} \lambda_t \rho = \sum_{t=2}^{T_0} \left(\pi^*(v_t)(r(v_t) - \lambda_t c(v_t)) + \lambda_t \rho\right).$$
(14)

Here (a) holds by the decision rule in Line 10; (b) follows from Lemma 3.2. Note that (b) brings no additional probability of failure as we already assume (4) and (5) holds for all t in the first step.

As π^* is optimal and feasible, we have

$$T \cdot \mathbb{E}_{v_t \sim F}[\pi^*(v_t)r(v_t)] = \text{OPT},$$
(15)

$$\mathbb{E}_{v_t \sim F}[\pi^*(v_t)c(v_t)] \le \rho. \tag{16}$$

Notice that λ_t and v_t are independent, so we further have for any t,

$$\mathbb{E}_{v_t \sim F} \left[\lambda_t \pi^*(v_t) c(v_t) - \lambda_t \rho \mid \mathcal{H}_t \right] \le 0.$$
(17)

Combining (15) and (17), we have

$$\mathbb{E}_{v_t \sim F} \left[\pi^*(v_t)(r(v_t) - \lambda_t c(v_t)) - \left(\frac{\text{OPT}}{T} - \lambda_t \rho\right) \middle| \mathcal{H}_t \right] \ge 0.$$

Thus, we can again apply Azuma–Hoeffding inequality and get

$$R_1 \ge \sum_{t=2}^{T_0} \left(\pi^*(v_t)(r(v_t) - \lambda_t c(v_t)) + \lambda_t \rho \right) \ge \frac{T_0 - 1}{T} \text{OPT} - \sqrt{2T \ln T},$$
(18)

As for the other part R_2 , with the help of Lemma 3.4, we have

$$R_2 \stackrel{(a)}{\geq} -\left(\frac{(1/\rho - 1)^2}{\eta_{T_0}} + \sum_{t=2}^{T_0} \eta_t\right) \stackrel{(b)}{\geq} -3(1/\rho - 1)\sqrt{T},\tag{19}$$

where (a) holds by setting $\lambda = 0$ in (6) and (b) holds since $\eta_t = (1/\rho - 1)/\sqrt{t}$ and $T_0 \leq T$. Applying (18) and (19) to (13), we have

$$\sum_{t=2}^{T_0} x_t \widetilde{r}_t(v_t) \ge \frac{T_0 - 1}{T} \text{OPT} - \sqrt{2T \ln T} - 3(1/\rho - 1)\sqrt{T},$$
(20)

which holds with an extra failure probability 1/T.

Step 3: bounding the stopping time T_0 . This step aims to show that Algorithm 1 does not run out of budget too early so that T_0 is actually close to T with high probability. Intuitively, we prove this by arguing that the expenditure rate of Algorithm 1 is fairly close to the target rate ρ . By the update rule we have for every $t \in \{2, \dots, T_0\}$,

$$\lambda_{t+1} \ge \lambda_t + \eta_t (x_t \widetilde{c}_t(v_t) - \rho).$$

Reordering terms and summing up from 2 through T_0 , one has

$$\sum_{t=2}^{T_0} \left(x_t \widetilde{c}_t(v_t) - \rho \right) \le \sum_{t=2}^{T_0} \frac{\lambda_{t+1} - \lambda_t}{\eta_t} \stackrel{\text{(a)}}{=} \frac{\lambda_{T_0+1}}{\eta_{T_0}} + \sum_{t=3}^{T_0} \left(\frac{1}{\eta_{t-1}} - \frac{1}{\eta_t} \right) \lambda_t$$

$$\stackrel{\text{(b)}}{\leq} \frac{\lambda_{T_0+1}}{\eta_T} \stackrel{\text{(c)}}{\leq} \frac{1/\rho - 1}{\eta_T}.$$
(21)

where (a) holds since $\lambda_2 = 0$; (b) uses the uses the construction that η_t is decreasing; and (c) holds by Lemma 3.3. Consequently, we have

$$\rho(T - T_0 + 1) \stackrel{\text{(a)}}{\leq} \sum_{t=2}^{T_0} x_t c_t + 2 - \rho (T_0 - 1)$$

$$\stackrel{\text{(b)}}{\leq} \sum_{t=2}^{T_0} x_t \widetilde{c}_t (v_t) - \rho (T_0 - 1) + \sqrt{2T \ln T} + 4 \sum_{t=2}^{T_0} x_t \epsilon_t v_t + 2$$

$$\stackrel{\text{(c)}}{\leq} \sum_{t=2}^{T_0} (x_t \widetilde{c}_t (v_t) - \rho) + \sqrt{2T \ln T} + 4 \sqrt{2T (\ln 2 + 2 \ln T)} + 2$$

$$\stackrel{\text{(d)}}{\leq} \sqrt{T} + \sqrt{2T \ln T} + 4 \sqrt{2T (\ln 2 + 2 \ln T)} + 2.$$

Here, (a) holds since $\sum_{t=2}^{T_0} x_t c_t + 2 \ge B = \rho T$; (b) is derived by applying the inequality (5) and Azuma-Hoeffding inequality to bound the difference between $\sum_{t=2}^{T_0} x_t c_t$ and $\sum_{t=2}^{T_0} x_t \tilde{c}_t(v_t)$; (c) follows from (11); and (d) follows from (21) and $\eta_T = (1/\rho - 1)/\sqrt{T}$. Note that only the Azuma-Hoeffding inequality in (b) brings an extra failure probability of 1/T.

Simple rearrangements establish that,

$$\frac{T_0 - 1}{T} \text{OPT} \ge \text{OPT} - \frac{\text{OPT}}{T\rho} \left(\sqrt{T} + \sqrt{2T \ln T} + 4\sqrt{2T(\ln 2 + 2\ln T)} + 2 \right).$$
(22)

Step 4: Putting everything together. Observe that $OPT/T = \Theta(1)$ and $\rho = \Theta(1)$. Therefore, combining inequalities (12), (20) and (22), there is a constant $C^{\rm F}$ such that with probability at least 1 - 4/T,

$$U^{\beta}(\boldsymbol{v}, \boldsymbol{p}) \ge \text{OPT} - C^{\text{F}} \sqrt{T \ln T},$$

which concludes the proof of Theorem 3.1.

B.5 Proof of Lemma 3.7

We demonstrate by analyzing the dynamic process of λ_t . Intuitively, the trajectory of λ_t can be divided into consecutive phases, based on whether $\lambda_t = 0$ or $\lambda_t > 0$.

Specifically, we prove by induction. When t = 2, the lemma naturally holds since Algorithm 1 chooses to enter in the first round, and therefore $|\mathcal{I}_2| = 1$.

Now suppose the lemma holds for all $2 \le t \le \tau$. When $t = \tau + 1$, we consider by cases on whether $\lambda_{\tau} = 0$.

Case 1: $\lambda_{\tau} = 0$. This case is rather trivial. Notice that by definition $\tilde{r}_{\tau}(v_{\tau}) \ge \epsilon_{\tau} v_{\tau} > 0$, therefore, $\tilde{r}_{\tau}(v_{\tau}) - \lambda_{\tau} \tilde{c}_{\tau}(v_{\tau}) > 0$ and $x_{\tau} = 1$ by the decision rule. As a result,

$$|\mathcal{I}_{\tau+1}| = |\mathcal{I}_{\tau}| + 1 \ge C_e(\tau - 1) + 1 \ge C_e\tau.$$

Case 2: $\lambda_{\tau} > 0$. In this case, we define $l(\tau) := \max\{t < \tau : \lambda_t = 0\}$ be the last time $t < \tau$ such that $\lambda_t = 0$. Note that such l(t) is well-defined since $\lambda_2 = 0$. We further define $\tau^+ := |\{l(\tau) \le t < \tau : x_t = 1\}|$ be the rounds in $[l(\tau), \tau)$ such that Algorithm 1 chooses to enter. We now give a lower bound on $\tau^+/(\tau - l(\tau) + 1)$.

Notice that by the definition of $l(\tau)$, we have $\lambda_{t+1} - \lambda_t = \eta_t (x_t \tilde{c}_t(v_t) - \rho)$ for any $l(\tau) \le t < \tau$. As a result, we have

$$\lambda_{\tau} - \lambda_{l(\tau)} = \sum_{t=l(\tau)}^{\tau-1} (\lambda_{t+1} - \lambda_t) = \sum_{t=l(\tau)}^{\tau-1} \eta_t (x_t \tilde{c}_t(v_t) - \rho)$$

$$\stackrel{(a)}{\leq} (1-\rho) \sum_{t=l(\tau)}^{l(\tau)+\tau^+-1} \eta_t - \rho \sum_{t=l(\tau)+\tau^+}^{\tau-1} \eta_t = \sum_{t=l(\tau)}^{l(\tau)+\tau^+-1} \eta_t - \rho \sum_{t=l(\tau)}^{\tau-1} \eta_t.$$

Here, (a) holds since when $x_t = 1$, $x_t \tilde{c}_t(v_t) - \rho \leq 1 - \rho$. Further, η_t is decreasing in t. Now that we have $\lambda_\tau > 0 = \lambda_{l(\tau)}$. Meanwhile, for any $2 \leq t_1 \leq t_2$, we have

$$2(\sqrt{t_2+1} - \sqrt{t_1}) \le \sum_{t=t_1}^{t_2} 1/\sqrt{t} \le 2(\sqrt{t_2} - \sqrt{t_1-1}).$$

Therefore, noticing that $\eta_t = (1/\rho - 1)/\sqrt{t}$, we have

$$0 \leq \lambda_{\tau} - \lambda_{l(\tau)} \leq (1-\rho) \sum_{t=l(\tau)}^{l(\tau)+\tau^{+}-1} \eta_{t} - \rho \sum_{t=l(\tau)+\tau^{+}}^{\tau-1} \eta_{t} = \sum_{t=l(\tau)}^{l(\tau)+\tau^{+}-1} \eta_{t} - \rho \sum_{t=l(\tau)}^{\tau-1} \eta_{t}$$
$$\leq 2(\sqrt{l(\tau)+\tau^{+}-1} - \sqrt{l(\tau)-1}) - 2\rho(\sqrt{\tau} - \sqrt{l(\tau)}).$$

That is, $\sqrt{l(\tau) + \tau^+ - 1} - \sqrt{l(\tau) - 1} \ge \rho(\sqrt{\tau} - \sqrt{l(\tau)})$. We temporarily write $u := \sqrt{\tau} - \sqrt{l(\tau)}$ for simplicity, and consequently derive that

$$\frac{\tau^{+}}{\tau - l(\tau) + 1} \geq \frac{(\rho u + \sqrt{l(\tau) - 1})^{2} - (l(\tau) - 1)}{(u + \sqrt{l(\tau)})^{2} - l(\tau)} \cdot \frac{\tau - l(\tau)}{\tau - l(\tau) + 1}$$

$$\stackrel{(a)}{\geq} \frac{1}{2} \cdot \frac{\rho^{2} u^{2} + 2\rho\sqrt{l(\tau) - 1}}{u^{2} + 2u\sqrt{l(\tau)}} \stackrel{(b)}{\geq} \min\left\{\frac{\sqrt{2}}{2}\rho^{2}, \frac{\sqrt{2}}{4}\rho\right\} = C_{e}.$$

Here, (a) establishes as $\tau - l(\tau) \ge 1$, and (b) holds since $l(\tau) \ge 2$. Hence, we derive that

$$|\mathcal{I}_{\tau+1}| \ge |\mathcal{I}_{l(\tau)}| + \tau^+ \ge C_e(l(\tau) - 1) + C_e(\tau - l(\tau) + 1) = C_e \tau.$$

By combining the two cases and induction, we conclude that Lemma 3.7 holds.

B.6 Proof of Theorem 3.5

According to Lemma 3.7, we have

$$\epsilon_t = \sqrt{\frac{\ln 2 + 2\ln T}{2|\mathcal{I}_t|}} \le \frac{\ln 2 + 2\ln T}{2C_e(t-1)}.$$

Hence,

$$\sum_{t=2}^{T_0} \epsilon_t \le \sum_{t=2}^{T_0} \sqrt{\frac{\ln 2 + 2\ln T}{2C_e(t-1)}} \le \sqrt{\frac{2T(\ln 2 + 2\ln T)}{C_e}}$$

Meanwhile, the failure probability in estimation is also bounded:

$$\sum_{t=2}^{T_0} q'(t, \epsilon_t) = 2 \sum_{t=2}^{T_0} \exp(-2|\mathcal{I}_t|\epsilon_t^2) \le 1/T.$$

The remaining proof of Theorem 3.5 is almost identical to the proof of Theorem 3.1, and we omit it.

C Missing Proofs in Section 4

C.1 Proof of Theorem 4.1

We construct such a problem instance with v always equal to 1, G be the uniform binary distribution on 1/3 and 2/3, and $\rho = 1/2$. Consider the optimal strategy π^* for OPT. Under this instance, we have $\mathbb{E}_{p\sim G}[p] = 1/2$. Therefore, $\pi^*(1/3) = \pi^*(2/3) = 1$, and OPT $= \theta^* \cdot \mathbb{E}_{p\sim G}[1-p] = 1/2$.

Now, we consider any online throttling strategy β when 4|T. Since v always equals to 1^T , we simplify the notation of $U^{\beta}(v, p)$ to $U^{\beta}(p)$ in context of the given instance. We give an upper bound on $U^{\beta}(p)$ in the following lemma, the proof of which can be found in Appendix D.1.

Lemma C.1. Given a realization of $\mathbf{p} \sim G^T$, we let S be the number of 1/3(s) in \mathbf{p} , then

$$U^{\beta}(\boldsymbol{p}) \leq \begin{cases} \frac{1}{3} \cdot (S+T) & \text{if } S \geq T/2, \\ \frac{2}{3} \cdot S + \frac{1}{3} \cdot \lfloor \frac{3T-2S}{4} \rfloor & \text{if } S < T/2. \end{cases}$$

With Lemma C.1, we can come to analyze the lower bound of the regret of any online throttling strategy. We first give a non-simplified formula on the regret with the following lemma, the proof of which can be found in Appendix D.2.

Lemma C.2. In the given instance (F, G, ρ) , for any strategy β and number of rounds T, we have

OPT
$$-\mathbb{E}_{\boldsymbol{p}\sim G^T}\left[U^{\beta}(\boldsymbol{p})\right] \ge \frac{1}{12\cdot 2^T} \sum_{t=1}^{T/4} \left(T - 4(t-1)\right) \binom{T+1}{2t-1}.$$

Now, we need to simplify the lower bound we give above, which we finish in the following lemma. In fact, the complex summing formula turns out to be much concise.

Lemma C.3.

$$\sum_{t=1}^{T/4} \left(T - 4(t-1)\right) \binom{T+1}{2t-1} = 2^{T-1} + \frac{T}{2} \binom{T}{T/2}$$

The proof of Lemma C.3 can be found in Appendix D.3.

At last, by combining Lemma C.2 and Lemma C.3 and noticing that $\binom{T}{T/2} \geq 2^T/\sqrt{2T}$ by Stirling's formula, we have

$$\mathbb{E}_{\boldsymbol{p}\sim G^{T}}\left[Reg^{\beta}(\boldsymbol{p})\right] = OPT - \mathbb{E}_{\boldsymbol{p}\sim G^{T}}\left[U^{\beta}(\boldsymbol{p})\right]$$

$$\geq \frac{1}{12 \cdot 2^T} \left(2^{T-1} + \frac{T}{2} \begin{pmatrix} T \\ T/2 \end{pmatrix} \right)$$
$$\geq \frac{1}{24} + \frac{\sqrt{2}}{48} \sqrt{T}.$$

Therefore, Theorem 4.1 holds with $C_l = \sqrt{2}/48$.

D Missing Proofs in the Appendix

D.1 Proof of Lemma C.1

Given p including S 1/3(s) and (T-S) 2/3(s), we let $n_1 \leq S$ and $n_2 \leq T-S$ be the better number of auctions in which the buyer enters with p = 1/3 and p = 2/3 respectively. Then by the budget constraint, we have

$$\frac{1}{3} \cdot n_1 + \frac{2}{3} \cdot n_2 \le \frac{1}{2} \cdot T.$$

Meanwhile, the total revenue gained by β under p is

$$U^{\beta}(\boldsymbol{p}) = \frac{2}{3} \cdot n_1 + \frac{1}{3} \cdot n_2.$$

When $S \ge T/2$, then since $n_1 \le S$ and $n_2 \le T - S$, it is certain that

$$U^{\beta}(\mathbf{p}) \le \frac{2}{3} \cdot S + \frac{1}{3} \cdot (T - S) = \frac{1}{3} \cdot (S + T).$$

When S < T/2, if S is even, then

$$U^{\beta}(\boldsymbol{p}) = \frac{2}{3} \cdot n_1 + \frac{1}{3} \cdot n_2 = \frac{1}{2}n_1 + \frac{1}{2}\left(\frac{1}{3} \cdot n_1 + \frac{2}{3} \cdot n_2\right)$$
$$\leq \frac{1}{2}S + \frac{1}{4}T,$$

and the equality holds with $n_1 = S$ and $n_2 = (3T - 2S)/12$. On the other hand, if S is odd, then by a similar argument the maximal value $U^{\beta}(\mathbf{p})$ can reach with $n_1 \leq S - 1$ is no larger than (S-1)/2 + T/4. If $n_1 = S$, then

$$U^{\beta}(\boldsymbol{p}) = \frac{2}{3} \cdot n_1 + \frac{1}{3} \cdot n_2 \le \frac{2}{3} \cdot S + \frac{1}{3} \left\lfloor \frac{T/2 - S/3}{2/3} \right\rfloor = \frac{3T + 6S - 2}{12}.$$

Putting together, we have $U^{\beta}(\mathbf{p}) \leq (3T + 6S - 2)/12$. Combining both cases on whether 2|S or not, we derive that

$$U^{\beta}(\boldsymbol{p}) \leq rac{2}{3} \cdot S + rac{1}{3} \cdot \left[rac{3T - 2S}{4}
ight].$$

D.2 Proof of Lemma C.2

We first calculate an upper bound on the expected total revenue of online throttling. Specifically, since 4|T, we have

$$U^{\beta}(\boldsymbol{p}) \stackrel{(a)}{=} \frac{1}{2^{T}} \left(\sum_{S=0}^{T/2-1} \left(\frac{2}{3}S + \frac{1}{3} \left\lfloor \frac{3T-2S}{4} \right\rfloor \right) \binom{T}{S} + \sum_{S=T/2}^{T} \frac{1}{3} (S+T) \binom{T}{S} \right)$$

$$\stackrel{\text{(b)}}{=} \frac{1}{2^T} \left(\sum_{S=0}^{T/2-1} \left(\frac{2}{3}T + \frac{1}{3}S + \frac{1}{3} \left\lfloor \frac{3T-2S}{4} \right\rfloor \right) \binom{T}{S} + \frac{T}{2} \binom{T}{T/2} \right).$$

Here, (a) is by Lemma C.1, and (b) holds by noticing that $\binom{T}{S} = \binom{T}{T-S}$. Now notice that

$$OPT = \frac{T}{2} = \frac{1}{2^T} \sum_{S=0}^T \frac{T}{2} {T \choose S} = \frac{1}{2^T} \left(\sum_{S=0}^{T/2-1} T {T \choose S} + \frac{T}{2} {T \choose T/2} \right),$$

and therefore, we derive that

$$\begin{aligned} \text{OPT} - U^{\beta}(\boldsymbol{p}) &\geq \frac{1}{2^{T}} \sum_{S=0}^{T/2-1} \frac{1}{3} \left(T - S - \left\lfloor \frac{3T - 2S}{4} \right\rfloor \right) \begin{pmatrix} T\\ S \end{pmatrix} \\ &\stackrel{\text{(a)}}{=} \frac{1}{3 \cdot 2^{T}} \sum_{t=1}^{T/4} \left(\frac{T}{4} - t + 1 \right) \left(\begin{pmatrix} T\\ 2t - 2 \end{pmatrix} + \begin{pmatrix} T\\ 2t - 1 \end{pmatrix} \right) \\ &\stackrel{\text{(b)}}{=} \frac{1}{12 \cdot 2^{T}} \sum_{t=1}^{T/4} \left(T - 4(t - 1) \right) \begin{pmatrix} T + 1\\ 2t - 1 \end{pmatrix}. \end{aligned}$$

Here, (a) holds by noticing that the multipliers of both $\binom{T}{2t-2}$ and $\binom{T}{2t-1}$ in the summing part equal to T/4 - t + 1. Meanwhile, (b) establishes since $\binom{T}{2t-2} + \binom{T}{2t-1} = \binom{T+1}{2t-1}$.

D.3 Proof of lemma C.3

To start with, by binomial theorem, we have

$$\sum_{t=1}^{T/2} {T \choose 2t-1} = \sum_{t=1}^{T/2+1} {T \choose 2t-2} = 2^{T-1},$$

by adopting $\binom{T}{t} = \binom{T}{T-t}$ again, we obtain

$$\sum_{t=1}^{T/4} \binom{T}{2t-1} = 2^{T-2}, \quad \sum_{t=1}^{T/4} \binom{T}{2t-2} = 2^{T-2} - \frac{1}{2} \binom{T}{T/2}.$$
(23)

With the help of these two equalities, we have the following computation:

$$\begin{split} \sum_{t=1}^{T/4} \left(T - 4(t-1)\right) \begin{pmatrix} T+1\\ 2t-1 \end{pmatrix} &= (T+2) \sum_{t=1}^{T/4} \begin{pmatrix} T+1\\ 2t-1 \end{pmatrix} - 2 \sum_{t=1}^{T/4} (2t-1) \begin{pmatrix} T+1\\ 2t-1 \end{pmatrix} \\ &\stackrel{\text{(a)}}{=} \left(T+2\right) \sum_{t=1}^{T/4} \left(\begin{pmatrix} T\\ 2t-1 \end{pmatrix} + \begin{pmatrix} T\\ 2t-2 \end{pmatrix} \right) - 2(T+1) \sum_{t=1}^{T/4} \begin{pmatrix} T\\ 2t-2 \end{pmatrix} \\ &\stackrel{\text{(b)}}{=} \left(T+2\right) \sum_{t=1}^{T/4} \begin{pmatrix} T\\ 2t-1 \end{pmatrix} - T \sum_{t=1}^{T/4} \begin{pmatrix} T\\ 2t-2 \end{pmatrix} \\ &= 2^{T-1} + \frac{T}{2} \begin{pmatrix} T\\ T/2 \end{pmatrix}. \end{split}$$

For the above, (a) is by $(2t-1)\binom{T+1}{2t-1} = (T+1)\binom{T}{2t-2}$, and (b) is derived by (23).

References

- Vibhanshu Abhishek and Kartik Hosanagar. Optimal bidding in multi-item multislot sponsored search auctions. Oper. Res., 61(4):855–873, 2013.
- [2] Jason Acimovic and Vivek F Farias. The fulfillment-optimization problem. In Operations Research & Management Science in the age of analytics, pages 218–237. INFORMS, 2019.
- [3] Jason Acimovic and Stephen C. Graves. Making better fulfillment decisions on the fly in an online retail environment. *Manuf. Serv. Oper. Manag.*, 17(1):34–51, 2015.
- [4] Deepak Agarwal, Souvik Ghosh, Kai Wei, and Siyu You. Budget pacing for targeted online advertisements at linkedin. In Sofus A. Macskassy, Claudia Perlich, Jure Leskovec, Wei Wang, and Rayid Ghani, editors, *The 20th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, KDD '14, New York, NY, USA - August 24 - 27, 2014*, pages 1613–1619. ACM, 2014.
- [5] Shipra Agrawal and Nikhil R. Devanur. Fast algorithms for online stochastic convex programming. In Piotr Indyk, editor, Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015, pages 1405– 1424. SIAM, 2015.
- [6] Shipra Agrawal and Nikhil R. Devanur. Linear contextual bandits with knapsacks. In Daniel D. Lee, Masashi Sugiyama, Ulrike von Luxburg, Isabelle Guyon, and Roman Garnett, editors, Advances in Neural Information Processing Systems 29: Annual Conference on Neural Information Processing Systems 2016, December 5-10, 2016, Barcelona, Spain, pages 3450–3458, 2016.
- [7] Shipra Agrawal, Nikhil R. Devanur, and Lihong Li. An efficient algorithm for contextual bandits with knapsacks, and an extension to concave objectives. In Vitaly Feldman, Alexander Rakhlin, and Ohad Shamir, editors, *Proceedings of the 29th Conference on Learning Theory*, *COLT 2016, New York, USA, June 23-26, 2016*, volume 49 of *JMLR Workshop and Conference Proceedings*, pages 4–18. JMLR.org, 2016.
- [8] Shipra Agrawal, Zizhuo Wang, and Yinyu Ye. A dynamic near-optimal algorithm for online linear programming. Oper. Res., 62(4):876–890, 2014.
- [9] Alessandro Arlotto and Itai Gurvich. Uniformly bounded regret in the multisecretary problem. Stochastic Systems, 9(3):231–260, 2019.
- [10] Alessandro Arlotto and Xinchang Xie. Logarithmic regret in the dynamic and stochastic knapsack problem with equal rewards. *Stochastic Systems*, 10(2):170–191, 2020.
- [11] Ashwinkumar Badanidiyuru, John Langford, and Aleksandrs Slivkins. Resourceful contextual bandits. In Maria-Florina Balcan, Vitaly Feldman, and Csaba Szepesvári, editors, Proceedings of The 27th Conference on Learning Theory, COLT 2014, Barcelona, Spain, June 13-15, 2014, volume 35 of JMLR Workshop and Conference Proceedings, pages 1109–1134. JMLR.org, 2014.
- [12] Santiago Balseiro, Omar Besbes, and Dana Pizarro. Survey of dynamic resource constrained reward collection problems: Unified model and analysis. Available at SSRN 3963265, 2021.
- [13] Santiago R. Balseiro, Omar Besbes, and Gabriel Y. Weintraub. Repeated auctions with budgets in ad exchanges: Approximations and design. *Manag. Sci.*, 61(4):864–884, 2015.

- [14] Santiago R. Balseiro and Yonatan Gur. Learning in repeated auctions with budgets: Regret minimization and equilibrium. *Manag. Sci.*, 65(9):3952–3968, 2019.
- [15] Santiago R. Balseiro, Anthony Kim, Mohammad Mahdian, and Vahab S. Mirrokni. Budgetmanagement strategies in repeated auctions. Oper. Res., 69(3):859–876, 2021.
- [16] Santiago R. Balseiro, Christian Kroer, and Rachitesh Kumar. Contextual first-price auctions with budgets. CoRR, abs/2102.10476, 2021.
- [17] Santiago R. Balseiro, Haihao Lu, and Vahab S. Mirrokni. The best of many worlds: Dual mirror descent for online allocation problems. CoRR, abs/2011.10124, 2020.
- [18] Omar Besbes, Yash Kanoria, and Akshit Kumar. The multisecretary problem with many types. *CoRR*, abs/2205.09078, 2022.
- [19] Christian Borgs, Jennifer T. Chayes, Nicole Immorlica, Kamal Jain, Omid Etesami, and Mohammad Mahdian. Dynamics of bid optimization in online advertisement auctions. In Carey L. Williamson, Mary Ellen Zurko, Peter F. Patel-Schneider, and Prashant J. Shenoy, editors, *Proceedings of the 16th International Conference on World Wide Web, WWW 2007, Banff, Alberta, Canada, May 8-12, 2007*, pages 531–540. ACM, 2007.
- [20] Robert L. Bray. Does the multisecretary problem always have bounded regret? *CoRR*, abs/1912.08917, 2019.
- [21] Pornpawee Bumpensanti and He Wang. A re-solving heuristic with uniformly bounded loss for network revenue management. *Manag. Sci.*, 66(7):2993–3009, 2020.
- [22] Andrea Celli, Riccardo Colini-Baldeschi, Christian Kroer, and Eric Sodomka. The parity ray regularizer for pacing in auction markets. In Frédérique Laforest, Raphaël Troncy, Elena Simperl, Deepak Agarwal, Aristides Gionis, Ivan Herman, and Lionel Médini, editors, WWW '22: The ACM Web Conference 2022, Virtual Event, Lyon, France, April 25 - 29, 2022, pages 162–172. ACM, 2022.
- [23] Denis Xavier Charles, Deeparnab Chakrabarty, Max Chickering, Nikhil R. Devanur, and Lei Wang. Budget smoothing for internet ad auctions: a game theoretic approach. In Michael J. Kearns, R. Preston McAfee, and Éva Tardos, editors, *Proceedings of the fourteenth ACM Conference on Electronic Commerce, EC 2013, Philadelphia, PA, USA, June 16-20, 2013*, pages 163–180. ACM, 2013.
- [24] Xi Chen, Christian Kroer, and Rachitesh Kumar. Throttling equilibria in auction markets. In Michal Feldman, Hu Fu, and Inbal Talgam-Cohen, editors, Web and Internet Economics
 17th International Conference, WINE 2021, Potsdam, Germany, December 14-17, 2021, Proceedings, volume 13112 of Lecture Notes in Computer Science, page 551. Springer, 2021.
- [25] Zhaohua Chen, Xiaotie Deng, Jicheng Li, Chang Wang, and Mingwei Yang. Budgetconstrained auctions with unassured priors. CoRR, abs/2203.16816, 2022.
- [26] Vincent Conitzer, Christian Kroer, Debmalya Panigrahi, Okke Schrijvers, Eric Sodomka, Nicolás E. Stier Moses, and Chris Wilkens. Pacing equilibrium in first-price auction markets. In Anna Karlin, Nicole Immorlica, and Ramesh Johari, editors, Proceedings of the 2019 ACM Conference on Economics and Computation, EC 2019, Phoenix, AZ, USA, June 24-28, 2019, page 587. ACM, 2019.

- [27] Vincent Conitzer, Christian Kroer, Eric Sodomka, and Nicolás E. Stier Moses. Multiplicative pacing equilibria in auction markets. Oper. Res., 70(2):963–989, 2022.
- [28] Nikhil R. Devanur, Kamal Jain, Balasubramanian Sivan, and Christopher A. Wilkens. Near optimal online algorithms and fast approximation algorithms for resource allocation problems. J. ACM, 66(1):7:1–7:41, 2019.
- [29] Facebook. Your guide to facebook bid strategy. https://www.facebook.com/gms_hub/share/biddingstrate 2017. Accessed: 2022-06-28.
- [30] Jon Feldman, Aranyak Mehta, Vahab S. Mirrokni, and S. Muthukrishnan. Online stochastic matching: Beating 1-1/e. In 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2009, October 25-27, 2009, Atlanta, Georgia, USA, pages 117–126. IEEE Computer Society, 2009.
- [31] Joaquin Fernandez-Tapia, Olivier Guéant, and Jean-Michel Lasry. Optimal real-time bidding strategies. Applied Mathematics Research eXpress, 2017(1):142–183, 2017.
- [32] Jason Gaitonde, Yingkai Li, Bar Light, Brendan Lucier, and Aleksandrs Slivkins. Budget pacing in repeated auctions: Regret and efficiency without convergence. CoRR, abs/2205.08674, 2022.
- [33] Guillermo Gallego, Huseyin Topaloglu, et al. *Revenue management and pricing analytics*, volume 209. Springer, 2019.
- [34] Anupam Gupta and Marco Molinaro. How the experts algorithm can help solve lps online. Math. Oper. Res., 41(4):1404–1431, 2016.
- [35] MohammadTaghi Hajiaghayi and Max Springer. Analysis of a learning based algorithm for budget pacing. CoRR, abs/2205.13330, 2022.
- [36] Elad Hazan. Introduction to online convex optimization. Foundations and Trends (R) in Optimization, 2(3-4):157–325, 2016.
- [37] Stefanus Jasin and Sunil Kumar. A re-solving heuristic with bounded revenue loss for network revenue management with customer choice. *Math. Oper. Res.*, 37(2):313–345, 2012.
- [38] Stefanus Jasin and Amitabh Sinha. An lp-based correlated rounding scheme for multi-item ecommerce order fulfillment. *Oper. Res.*, 63(6):1336–1351, 2015.
- [39] Chinmay Karande, Aranyak Mehta, and Ramakrishnan Srikant. Optimizing budget constrained spend in search advertising. In Stefano Leonardi, Alessandro Panconesi, Paolo Ferragina, and Aristides Gionis, editors, Sixth ACM International Conference on Web Search and Data Mining, WSDM 2013, Rome, Italy, February 4-8, 2013, pages 697–706. ACM, 2013.
- [40] Richard M. Karp, Umesh V. Vazirani, and Vijay V. Vazirani. An optimal algorithm for on-line bipartite matching. In Harriet Ortiz, editor, *Proceedings of the 22nd Annual ACM Symposium* on Theory of Computing, May 13-17, 1990, Baltimore, Maryland, USA, pages 352–358. ACM, 1990.
- [41] Anton J. Kleywegt and Jason D. Papastavrou. The dynamic and stochastic knapsack problem. Oper. Res., 46(1):17–35, 1998.

- [42] Anton J. Kleywegt and Jason D. Papastavrou. The dynamic and stochastic knapsack problem with random sized items. Oper. Res., 49(1):26–41, 2001.
- [43] Kuang-Chih Lee, Ali Jalali, and Ali Dasdan. Real time bid optimization with smooth budget delivery in online advertising. In Proceedings of the seventh international workshop on data mining for online advertising, pages 1–9, 2013.
- [44] Xiaocheng Li, Chunlin Sun, and Yinyu Ye. Simple and fast algorithm for binary integer and online linear programming. In Hugo Larochelle, Marc'Aurelio Ranzato, Raia Hadsell, Maria-Florina Balcan, and Hsuan-Tien Lin, editors, Advances in Neural Information Processing Systems 33: Annual Conference on Neural Information Processing Systems 2020, NeurIPS 2020, December 6-12, 2020, virtual, 2020.
- [45] Pascal Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. The Annals of Probability, pages 1269–1283, 1990.
- [46] Aranyak Mehta et al. Online matching and ad allocation. Foundations and Trends in Theoretical Computer Science, 8(4):265–368, 2013.
- [47] Aranyak Mehta, Amin Saberi, Umesh V. Vazirani, and Vijay V. Vazirani. Adwords and generalized online matching. J. ACM, 54(5):22, 2007.
- [48] Rui Sun, Xinshang Wang, and Zijie Zhou. Near-optimal primal-dual algorithms for quantitybased network revenue management. arXiv preprint arXiv:2011.06327, 2020.
- [49] Alberto Vera and Siddhartha Banerjee. The bayesian prophet: A low-regret framework for online decision making. *Manag. Sci.*, 67(3):1368–1391, 2021.
- [50] Huasen Wu, R. Srikant, Xin Liu, and Chong Jiang. Algorithms with logarithmic or sublinear regret for constrained contextual bandits. In Corinna Cortes, Neil D. Lawrence, Daniel D. Lee, Masashi Sugiyama, and Roman Garnett, editors, Advances in Neural Information Processing Systems 28: Annual Conference on Neural Information Processing Systems 2015, December 7-12, 2015, Montreal, Quebec, Canada, pages 433–441, 2015.
- [51] Jian Xu, Kuang-chih Lee, Wentong Li, Hang Qi, and Quan Lu. Smart pacing for effective online ad campaign optimization. In Longbing Cao, Chengqi Zhang, Thorsten Joachims, Geoffrey I. Webb, Dragos D. Margineantu, and Graham Williams, editors, Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, Sydney, NSW, Australia, August 10-13, 2015, pages 2217–2226. ACM, 2015.