Envy-free division of multi-layered cakes

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Thanks!

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Figure 1: Authors

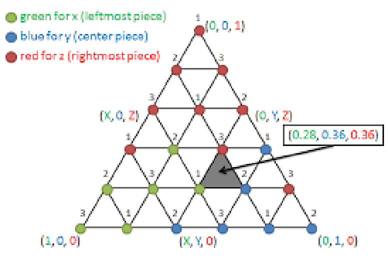
Introduction

The story

- n researchers allocate m meeting rooms over a period of time
- different opinions about the time slot and facility to which they would like to be assigned
- reduce to the one-dimensional case ? **failed**
- **feasible**: overlapping time intervals to the same agent who can utilize at most one room at a given time
- **the multi-layered cake-cutting problem** : n agents divide m different cakes under the **feasibility** constraint: the pieces of different layers assigned to the same agent should be non-overlapping, i.e., these pieces should have disjoint interiors.

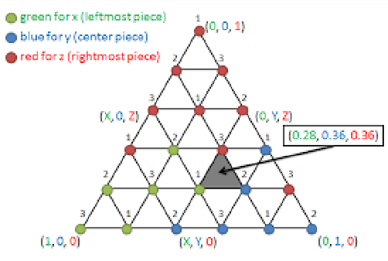
Envy-free division of 1-layer cake using Sperner's Lemma

- the divisions into n parallel pieces of lengths $x_i (i = 1, 2, ..., n)$ can be represented by the points of the standard simplex Δ_{n-1}
- vertices of each triangle being labeled as distinct owner agents
- each "owner" agent colors the vertex with the index of their most favorite bundle of the "owned" division



Envy-free division of 1-layer cake using Sperner's Lemma

- agents always prefer non-degenerate pieces to the degenerate one
- a colorful triangle, which corresponds to an approximate envy-free division, is guaranteed to exist by Sperner's lemma



Key idea

Encode a division with a configuration point $\mathbf{x} \in \|\Delta_{2n-1,n}\|$,

then **label** all points in configurations space,

finally **converge** to a fully labeled point (envy-free division)

Preliminaries

Definitions

- m layers and n agents
- A cake : unit interval $[0, 1]$
- A piece of cake : a union of finitely many disjoint closed subintervals of $[0, 1]$
- A contiguous piece of cakea : a subinterval of $[0, 1]$
- An m-layered cake : a sequence of m cakes $[0, 1]$
- A layered piece : a sequence $\mathcal{L} = (L_l)l \in [m]$ of pieces of each layer l
- a layered piece is contiguous if each L_l is a contiguous piece of each layer
- A layered piece $\mathcal{L}$ is non-overlapping if no two pieces from different layers overlap

Definitions

- A layered piece $\mathcal{L}$ is of zero-length if the piece of each layer is of zero-length
- A multi-division $\mathcal{A} = (\mathcal{A}_1, \cdots, \mathcal{A}_n)$ is a partition of the m-layered cake into n layered pieces
- A multi-division $\mathcal{A}$ is
  - contiguous if $\mathcal{A}_i$ is contiguous for each $i \in [n]$
  - feasible if $\mathcal{A}_i$ is non-overlapping for each $i \in [n]$
- Each agent i has a choice function c_i that, given several layered pieces, selects the preferred layered pieces
- An agent has hungry preferences if she weakly prefers a layered piece of nonzero length to any layered piece of zero-length

Definitions

An agent has *monotone preferences* if she always prefers a layered piece $\mathcal{L} = (L_\ell)_{\ell \in [m]}$ to another $\mathcal{L}' = (L'_\ell)_{\ell \in [m]}$ whenever L'_ℓ is contained in L_ℓ for each $\ell \in [m]$.

An agent has *closed preferences* if the following holds: for every sequence $(\mathcal{A}^{(t)})_{t \in \mathbb{Z}_+}$ of multi-divisions converging to a multi-division $\mathcal{A}^{(\infty)}$, we have

$$\mathcal{A}_j^{(t)} \in c_i(\mathcal{A}^{(t)}) \quad \forall t \in \mathbb{Z}_+ \qquad \Longrightarrow \qquad \mathcal{A}_j^{(\infty)} \in c_i(\mathcal{A}^{(\infty)})\,.$$

The convergence of layered pieces is considered according to the pseudo-metric $d(\mathcal{L}, \mathcal{L}') = \mu(\mathcal{L} \triangle \mathcal{L}')$; a sequence of multi-divisions is converging if each of its layered pieces converges. Here, μ is the Lebesgue measure and $\mathcal{L} \triangle \mathcal{L}' = ((L_\ell \setminus L'_\ell) \cup (L'_\ell \setminus L_\ell))_{\ell \in [m]}$.

A multi-division $\mathcal{A}$ is *envy-free* if there exists a permutation $\pi \colon [n] \to [n]$ such that $\mathcal{A}_{\pi(i)} \in c_i(\mathcal{A})$ for all $i \in [n]$. A *birthday cake multi-division* for agent i^* is a multi-division $\mathcal{A}$ where no matter which piece agent i^* selects, there is an envy-free assignment of the remaining pieces to the remaining agents, i.e., for every $j \in [n]$, there exists a bijection $\pi \colon [n] \setminus \{i^*\} \to [n] \setminus \{j\}$ such that $\mathcal{A}_{\pi(i)} \in c_i(\mathcal{A})$ for all $i \in [n] \setminus \{i^*\}$.

Definitions

- Each agent i has a valuation function v_i that assigns a real value $v_i(\mathcal{L})$ to any layered piece $\mathcal{L}$
- v_i satisfies

  - *monotonicity* if $v_i(\mathcal{L}') \leqslant v_i(\mathcal{L})$ for any pair of layered pieces $\mathcal{L}, \mathcal{L}'$ such that $L'_\ell \subseteq L_\ell$ for any $\ell \in [m]$.
  - *the Lipschitz condition* if there exists a fixed constant K such that for any pair of layered pieces $\mathcal{L}, \mathcal{L}'$, $|v_i(\mathcal{L}) - v_i(\mathcal{L}')| \leqslant K \times \mu(\mathcal{L} \triangle \mathcal{L}')$.

- A multi-division A is ϵ-envy-free if different agents approximately prefer different layered pieces

$$\exists \pi : [n] \to [n], \forall i \in [n]$$

$$v_i(\mathcal{A}_{\pi(i)}) + \epsilon \geq \max_{i' \in [n]} v_i(\mathcal{A}_{i,})$$

long knives and short knives

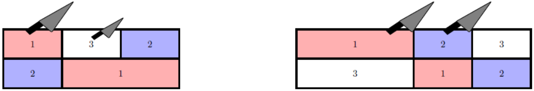


Fig. 1: Multi-divisions of a two-layered cake into three-layered pieces, obtained by one long knife and one short knife and by two long knives (pictured left-to-right).

Envy-free division using $n-1$ long knives

Theorem 1

Theorem 1. *Consider an instance of the multi-layered cake-cutting problem with m layers and n agents, $m \leqslant n$, with closed preferences. If n is a prime power, then there exists an envy-free feasible (i.e., non-overlapping) multi-division obtained by $n-1$ long knives.*

Chessboard complex

The chessboard complex $\Delta_{m,n}$ is an abstract simplicial complex whose ground set is $[m] \times [n]$

simplices are the subsets $\sigma \subset [m] \times [n]$ such that for every two distinct pairs (i, j) and (i', j') in σ we have $i \neq i'$ and $j \neq j'$

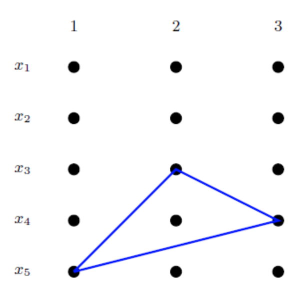


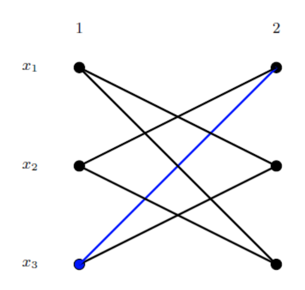
Figure 2: An simplex in complex $\Delta_{5,3}$

How to cut

- $n = p^k$ and $G = (\mathbb{Z}_p)^k$
- arbitrary bijection $\eta : [n] \to G$ (if $k = 1$ we can set $\eta(i) = i$)
- arbitrary injective map $h : \{layers\} \to G$
- $\Delta_{2n-1,n}$ and $\mathbf{v}_{i,j}$ as vertex (i, j)
- each point of $\|\Delta_{2n-1,n}\|$ encodes a multi-division with $n - 1$ long knives

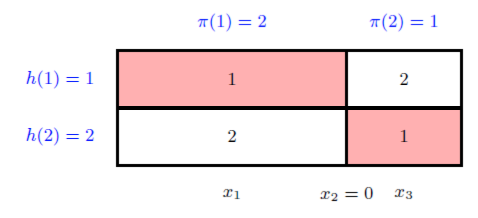
Encode multi-division with point of $\|\Delta_{2n-1,n}\|$

- $\mathbf{x} \in \|\Delta_{2n-1,n}\|$ arbitrary choose a simplex containing it
- the vertices of the simplex is $\mathbf{v}_{i_1,1}, \mathbf{v}_{i_2,2}, \cdots, \mathbf{v}_{i_n,i}$
- $\mathbf{x} = \sum_{j=1}^{n} x_{i_j} v_{i_j,j}$
- Choose permutation π that $i_{\pi(1)} < i_{\pi(2)} < \cdots < i_{\pi(n)}$



Encode multi-division with point of $\|\Delta_{2n-1,n}\|$

- $x_{i_{\pi(j)}}$ is the length of j-th piece in any layer
- bundle name of j-th piece in l-th layer is $\eta(\pi(j)) + h(l)$
- $\eta(\pi(j)) + h(l) = \eta(\pi(j)) + h(l')$ if and only if $l = l'$ so no overlap
- $\mathcal{A}(x) = (\mathcal{A}_1(x), \cdots, (A)_n(x))$,$\mathcal{A}_i(x)$ is pieces with bundle $\eta(i) \in G$



lemmas

Lemma 1. *Let $n = p^k$, where p is a prime number and k a positive integer. Denote by G the additive group $(\mathbb{Z}_p)^k$. Consider a G-invariant triangulation T of $\Delta_{2n-1,n}$. For any G-equivariant labeling of the vertices of T with elements of G, there is a simplex in T whose vertices are labeled with all elements of G.*

Lemma 2. *There exists a G-invariant triangulation T of $\Delta_{2n-1,n}$ of arbitrary small mesh size, refining $\Delta_{2n-1,n}$, and with an agent labeling $a\colon V(\mathsf{T}) \to \{\text{agents}\}$ that satisfies $a(g \cdot \boldsymbol{v}) = a(\boldsymbol{v})$ for all $g \in G$ and $\boldsymbol{v} \in V(\mathsf{T})$.*

Lemma 3. *Let $\left(\boldsymbol{x}^{(t)}\right)_{t\in\mathbb{Z}_+}$ be a sequence of points of $\|\Delta_{2n-1,n}\|$ converging to some limit point $\boldsymbol{x}^{(\infty)}$. Then $\left(\mathcal{A}(\boldsymbol{x}^{(t)})\right)_{t\in\mathbb{Z}_+}$ converges to $\mathcal{A}(\boldsymbol{x}^{(\infty)})$.*

Recall Theorem 1

Theorem 1. *Consider an instance of the multi-layered cake-cutting problem with m layers and n agents, $m \leqslant n$, with closed preferences. If n is a prime power, then there exists an envy-free feasible (i.e., non-overlapping) multi-division obtained by $n-1$ long knives.*

Proof of theorem 1

- triangulation T and an agent-labeling $a : V(T) \to \{agents\}$
- partition the vertices of T into their G-orbits
- From each orbit, we pick a vertex v. We ask agent $a(v)$ the index i of the non-overlapping layered piece $\mathcal{A}_i(x)$ she prefers in $\mathcal{A}(v)$.
- define $\lambda(v) = \eta(i)$
- extend λ on each orbit in an equivariant way:

$$\lambda(g \cdot v) := g \cdot \lambda(v)$$

unambiguously because the action of G on T is free(Sorry, I have not understood this)

Proof of theorem 1

- According to Lemma 1, there exists a simplex of T whose vertices are labeled with all elements of G
- $\forall N \in \mathbb{Z}_+, \exists T := T_N$ that mesh size upper bounded by 1/N(why mesh size $\rightarrow 0$?)
- Denote by $x^{i,N}$ the vertex of this simplex in TN that has $a(x^{i,N}) = i$
- $\pi_N(i)$ the integer $\eta^{-1}(\lambda(x^{i,N}))$
- $N \rightarrow \infty$, at least one permutation π_N occurring infinitely many times, denote it as π
- for these N that $\pi_N = \pi$, we have $\mathcal{A}_{\pi(i)}(x^{i,N}) \in c_i(\mathcal{A}(x^{i,N}))$ for all $i \in [n]$ and all such N. Select a infinite sequence from these N and $(x^{i,N})_N$ converges to x^* for all i, then $\mathcal{A}^* = \mathcal{A}(x^*)$ is the envy-free division

This paper also proved

Theorem 2. *In the case with two layers and three agents with closed, monotone, and hungry preferences, a birthday cake multi-division that is feasible and contiguous exists. Moreover, it requires only one long knife.*

Theorem 3. *In the case with two layers and three agents whose valuation functions satisfy the Lipschitz condition and monotonicity, for any $\varepsilon \in (0, 1)$, an ε'-birthday cake multi-division that is feasible and contiguous where $\varepsilon' \in O(\varepsilon)$ can be found in $O(\log^2 \frac{1}{\varepsilon})$ time. Moreover, it requires only one long knife.*

Algorithm 1: Computing an approximate envy-free multi-division

Initialize $X \leftarrow [0, 1]$ and $Y \leftarrow [0, 1]$;
while $\max X - \min X > 2\varepsilon$ **do**
  Set $X_1 = [\min X, \text{med } X]$ and $X_2 = [\text{med } X, \max X]$;
  Compute $\mathsf{d}(X_1 \cdot Y)$ and $\mathsf{d}(X_2 \cdot Y)$;
  Update $X \cdot Y$ to the one with nonzero degree;
while $\max Y - \min Y > 2\varepsilon$ **do**
  Set $Y_1 = [\min Y, \text{med } Y]$ and $Y_2 = [\text{med } Y, \max Y]$;
  Compute $\mathsf{d}(X \cdot Y_1)$ and $\mathsf{d}(X \cdot Y_2)$;
  Update $X \cdot Y$ to the one with nonzero degree;
return $X \cdot Y$;

Open questions

Open questions

- Even if $n \neq p^k$, p is a prime number, such that $n = 6$, the envy-free division is still not solved.
- This division of multi-layered cakes problem is formulated just in 2020 IJCAI and only a small setting is solved in this work.
- The problem itself is a big open question.

Thanks!